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Multi-component, concentric, copolar, axisymmetric, rigidly rotating polytropes: an improved and extended theory

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Abstract

With respect to earlier investigations, the theory of multi-component, concentric, copolar, axisymmetric, rigidly rotating polytropes is improved and extended, including subsystems with nonzero density on the boundary and subsystems with intersecting boundaries. The formulation is restricted to two subsystems for simplicity but, in principle, can be extended to N subsystems. Equilibrium configurations are independent of the nature of the fluid i.e. collisional or collisionless, provided the polytropic index lies within the range, $1/2 \leq n \leq 5$, as in one-component systems. The solution of the equilibrium equations is expanded in power series, which can be continued up to the boundary and outside via starting points placed at increasingly larger distance from the centre of mass. A detailed analysis is devoted to special cases where the solution of the equilibrium equations can be expressed analytically. Finally a guidance example is shown, involving homogeneous subsystems with intersecting boundaries, where a substantially flattened component extends outside a slightly flattened one.

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1 Introduction

Special attention has been devoted to polytropes since more than a century, and is continuing to be devoted at present. In fact, polytropic models are useful not only for the rough estimates of some processes in real stars, but also in the precise investigation of some matters of principle, such as the effect of increasing central condensation on nonradial pulsation models [26], the structure of supermassive and superdense stars [42], the collapse of stellar cores [17], the fission and the equatorial shedding of matter due to rotation [24], [23], [22], [30], [33], [28], [19], the oscillation and stability of rapidly rotating stellar models [38], [2], [3], [32], the structure of neutron stars and quark stars [16], [25], the expression of the total mass of a rotating configuration as a function of the central density [35], equilibrium configurations and triaxiality in stars and stellar systems [39], [40]. For exhaustive review and complete references, an interested reader is addressed to classical textbooks concerning stellar structure [14] and, in addition, applications in astrophysics and related fields [21]. Among others, polytropes can provide a continuous transition from null to infinite matter concentration i.e. homogeneous and Roche (mass points surrounded by vanishing atmospheres) models, distorted by rotation and/or tidal interaction [24], Chap. IX, §235.

On the other hand, large-scale stellar systems (galaxies and galaxy clusters) appear to be made of at least two subsystems interacting only via gravitation e.g., bulge+disk, core+halo, visible baryonic (including leptons)+dark non-baryonic matter. In most applications, each component is treated separately and the model is a simple superposition of the two matter distributions e.g., [12]. To this respect, models involving two-component polytropes make a further step in that each subsystem readjusts itself in presence of tidal interaction [10], [5], [7] and, in addition, both collisional and collisionless fluids can be described within the range of polytropic index, $1/2 \leq n \leq 5$, [39] where $n = 0, 5$, relates to homogeneous and Roche or Plummer models, respectively.

The current paper aims to improve and extend earlier investigation on two-component polytropes [10], [5], [7], mainly focusing on the following points: (i) formulation of the theory where due effort is devoted to the connection with one-component polytropes; (ii) extension of the theory to subsystems with nonzero density on the boundary; (iii) extension of the theory to subsystems with intersecting boundaries; (iv) extension of the theory to collisionless fluids; (v) series expansion of the solution of equilibrium equations, which can be continued up to the boundary and outside, via starting points placed at increasingly larger distance from the centre of mass; (vi) detailed analysis of

special cases where the solution of equilibrium equations can be expressed analytically; (vii) a guidance example involving homogeneous subsystems with intersecting boundaries, where a substantially flattened component extends outside a slightly flattened one.

The paper is organized as follows. Points (i)-(vi) mentioned above make the subject of Section 2, where each argument is discussed in one or more Subsections. The guidance example stated on point (vii) is shown in Section 3. The discussion is performed in Section 4 and the conclusion is drawn in Section 5. Further details on the formulation of the theory can be seen in the Appendix.

2 Basic theory

2.1 Collisional fluids

The theory of multi-component polytropes has been developed in earlier attempts [10], [5], [7] and shall not be repeated, unless extensions and improvements are involved. An interested reader is addressed to the above quoted parent papers. In any case, attention is restricted to concentric, copolar, axisymmetric, rigidly rotating polytropes which interact only via gravitation.

Accordingly, the pressure and the density within a generic subsystem, w , on a total of N , read:

$$p_w = K_w(\rho_w^{1+1/n_w} - \rho_{b,w}^{1+1/n_w}) \quad ; \quad w = i_1, i_2, \dots, i_N \quad ; \quad (1)$$

$$\rho_w = \lambda_w \theta_w^{n_w} \quad ; \quad w = i_1, i_2, \dots, i_N \quad ; \quad (2)$$

where K_w and the polytropic index, n_w , are two constants, $\rho_{b,w}$ is the density on the boundary, λ_w is the central density, $\theta_w^{n_w}$ is a reduced density, and the general case relates to nonzero density on the boundary e.g., [24], Chap. IX, §235, [41], [35].

Strictly speaking, K_w has no physical meaning as related dimensions would involve density to a real power. The combination of Eq. (1) with its counterpart particularized to the centre of mass yields:

$$\phi_w = \frac{\rho_w^{1+1/n_w} - \rho_{b,w}^{1+1/n_w}}{\lambda_w^{1+1/n_w} - \rho_{b,w}^{1+1/n_w}} = \frac{\phi_w^{1+1/n_w} - \phi_{b,w}^{1+1/n_w}}{1 - \phi_{b,w}^{1+1/n_w}} = \frac{\theta_w^{n_w+1} - \theta_{b,w}^{n_w+1}}{1 - \theta_{b,w}^{n_w+1}} \quad ; \quad (3)$$

$$\phi_w = \frac{p_w}{\pi_w} \quad ; \quad \phi_w = \frac{\rho_w}{\lambda_w} = \theta_w^{n_w} \quad ; \quad (4)$$

where π_w is the central pressure and ϕ , ϕ , are a reduced pressure and a reduced density, respectively.

Let a Cartesian reference frame have origin coinciding with the centre of mass of the whole system (which, in turn, coincides with the centre of mass

of each subsystem) and coordinate axes coinciding with the principal axes of inertia of the whole system (which, in turn, coincide with the principal axes of inertia of each subsystem). For reasons of simplicity, the centre of mass shall be hereafter quoted as the centre or the origin (of the reference frame), unless otherwise specified. Without loss of generality, let the polar axis coincide with the x_3 coordinate axis.

A necessary and sufficient condition for the w subsystem to be in equilibrium is [5], [7]:

$$\Delta\mathcal{V}_w = \Delta\mathcal{V}_G + 2\Omega_w^2 = -4\pi G\rho + 2\Omega_w^2 \quad ; \quad w = i_1, i_2, \dots, i_N \quad ; \quad (5)$$

where \mathcal{V}_w is the total (gravitational + centrifugal) potential on the w subsystem, \mathcal{V}_G is the gravitational potential induced by the whole system, Ω_w is the angular velocity of the w subsystem and $\rho = \rho_{i_1} + \rho_{i_2} + \dots + \rho_{i_N}$ is the local density of the whole system.

From this point on, attention shall be restricted to the simple case of two concentric, copolar, axisymmetric, rigidly rotating polytropes, hereafter quoted as EC2 polytropes, which may exhibit non intersecting or intersecting boundaries [5], [7]. Accordingly, the two subsystems may or may not lie one completely within the other. Let the two subsystems be denoted as $i_1 = i$; $i_2 = j$; respectively, and $u = i, j$; $v = j, i$; respectively, whenever they remain unspecified. Let a generic unspecified subsystem be denoted as $w = u, v$.

Let the volume where both subsystems coexist be defined as the common region, and the volume where only one is present be defined as the noncommon region. Without loss of generality, let the subsystem, i , exhibit the pole closer to the origin i.e. the shorter polar axis, and let i, j , be defined as the inner and the outer subsystem, respectively. Strictly speaking, the inner subsystem should be defined in connection with the undistorted configuration, $\Omega_i = \Omega_j = 0$, inferred from the knowledge of the initial conditions which, on the other hand, is not possible to ascertain for astrophysical systems.

The gravitational potential induced by the whole system on a generic internal point, $\mathbf{P}(x_1, x_2, x_3)$, is [5], [7]:

$$\mathcal{V}_G = K_w(n_w + 1)\lambda_w^{1/n_w}(\theta_w - \theta_{b,w}) - \frac{1}{2}\Omega_w^2(x_1^2 + x_2^2) + \mathcal{V}_{b,w} \quad ; \quad (6)$$

where $\theta_{b,w}^{n_w}$ is the reduced density, $\theta_w^{n_w}$, on the w boundary and $\mathcal{V}_{b,w}$ is the potential on the w boundary.

The second term on the right-hand side of Eq. (6) relates to the centrifugal potential, $\mathcal{V}_{C,w}$, which implies the remaining part relates to the total potential, $\mathcal{V}_w = \mathcal{V}_G + \mathcal{V}_{C,w}$, as:

$$\mathcal{V}_w = K_w(n_w + 1)\lambda_w^{1/n_w}(\theta_w - \theta_{b,w}) + \mathcal{V}_{b,w} \quad ; \quad (7)$$

and the equipotential surfaces coincide with both the isopycnic i.e. constant density surfaces and the isobaric surfaces [24], Chap. IX, §224.

The condition of centrifugal support on a generic point on the equatorial plane, $(\varpi, 0)$, reads:

$$\left(\frac{\partial \mathcal{V}_w}{\partial r}\right)_{\varpi,0} = \left(\frac{\partial \mathcal{V}_G}{\partial r}\right)_{\varpi,0} + \left(\frac{\partial \mathcal{V}_{C,w}}{\partial r}\right)_{\varpi,0} = 0 ; \quad (8)$$

where $\varpi = (x_1^2 + x_2^2)^{1/2}$. According to the above considerations, Eq. (8) via (6) reduces to:

$$\left(\frac{\partial \theta_w}{\partial r}\right)_{\varpi,0} = 0 ; \quad (9)$$

where positive values on the left-hand side cause instability in that centrifugal support is exceeded e.g., [10], [5], [7].

A necessary and sufficient condition for the w subsystem to be in equilibrium, Eq. (5), in the case under discussion, via Eqs. (2) and (6) reduces to:

$$K_w(n_w + 1)\lambda_w^{1/n_w} \Delta \theta_w - 2\Omega_w^2 = -4\pi G \sum [\lambda_w \theta_w^{n_w}] ; \quad (10)$$

where, in general, $\sum(F_w) = F_u + F_v$ and $\theta_v^{n_v} = 0$ in the noncommon region filled by u subsystem.

Let the following parameters be defined as [5], [7]:

$$\alpha_w = \Lambda_w^{1/2} \alpha_{w1} ; \quad v_w = \Lambda_w v_{w1} ; \quad \xi_w = \Lambda_w^{-1/2} \xi_{w1} ; \quad \Lambda_w = \frac{\lambda_w}{\sum(\lambda_w)} ; \quad (11)$$

where α is a scaling radius, v a rotation parameter, Λ_w a central density ratio, ξ a scaled radial coordinate, and the index, $w1$, denotes a one-component polytrope, hereafter quoted as EC1 polytrope, defined by the w subsystem i.e. the limit of a vanishing (other than w) subsystem. Accordingly, related explicit expressions read e.g., [4]:

$$\alpha_{w1} = \left[\frac{(n_w + 1)K_w \lambda_w^{1/n_w}}{4\pi G \lambda_w} \right]^{1/2} ; \quad v_{w1} = \frac{\Omega_w^2}{2\pi G \lambda_w} ; \quad \xi_{w1} = \frac{r}{\alpha_{w1}} ; \quad (12)$$

where $K_w \lambda_w^{1+1/n_w}$ is dimensioned as a pressure and r is the usual radial coordinate.

The additional relations:

$$\alpha_{v1} \xi_{v1} = \alpha_v \xi_v = \alpha_u \xi_u = \alpha_{u1} \xi_{u1} ; \quad (13)$$

$$\alpha_{v1} \Xi_{v1} = \alpha_v \Xi_v = \alpha_u \xi_u^* = \alpha_{u1} \xi_{u1}^* ; \quad (14)$$

$$\sum(\Lambda_w) = 1 ; \quad (15)$$

$$\sum(\lambda_w) \alpha_w^2 = \lambda_w \alpha_{w1}^2 ; \quad (16)$$

$$v_w \xi_w^2 = v_{w1} \xi_{w1}^2 ; \quad (17)$$

follow from Eq. (11) via (12), where Ξ_v , ξ_u^* , define the interface provided $\alpha_v \Xi_v(\mu) < \alpha_u \Xi_u(\mu)$ along the direction considered. It is worth noticing $\xi_u^* = \xi_u^*(\mu)$ does not imply $\theta_u(\xi_u^*) = \text{const}$, unless the two subsystems rotate to the same extent, $v_u = v_v$.

The substitution of Eq. (11), (12), into (6) and (10), after replacing Cartesian with polar coordinates, yields [5], [7]:

$$\mathcal{V}_G = 4\pi G \sum (\lambda_w) \alpha_w^2 \left\{ \theta_w - \theta_{b,w} - \frac{1}{6} v_w \xi_w^2 [1 - P_2(\mu)] \right\} + \mathcal{V}_{b,w} ; \quad (18)$$

$$\Delta \theta_w - v_w = - \sum (\Lambda_w \theta_w^{n_w}) ; \quad (19)$$

where $(r, \mu) = (\alpha_w \xi_w, \cos \delta)$ are polar coordinates (δ polar angle), implying $x_1^2 + x_2^2 = \alpha_w^2 \xi_w^2 (2/3) [1 - P_2(\mu)]$, $P_2(\mu) = (3\mu^2 - 1)/2$ is the Legendre polynomial of degree, 2, and a vanishing subsystem in the non common region implies a null density, $\theta_v^{n_v}(\xi_v, \mu) = 0$, $\xi_v \geq \Xi_v$, along the direction considered, which makes the sum on the right-hand side of Eq. (19) reduce to a single term, $\Lambda_u \theta_u^{n_u}(\xi_u, \mu)$, along the direction considered.

In terms of the parameters, α_{w1} , v_{w1} , and the variable, ξ_{w1} , Eq. (18) via (16) and (17) translates into:

$$\mathcal{V}_G = 4\pi G \lambda_w \alpha_{w1}^2 \left\{ \theta_w - \theta_{b,w} - \frac{1}{6} v_{w1} \xi_{w1}^2 [1 - P_2(\mu)] \right\} + \mathcal{V}_{b,w} ; \quad (20)$$

which is the formal expression of the gravitational potential of EC1 polytropes related to central density, λ_w , scaling radius, α_{w1} , and rotation parameter, v_{w1} .

In dimensionless polar coordinates, the Laplace operator translates into:

$$\Delta = \frac{1}{\xi_w^2} \frac{\partial}{\partial \xi_w} \left(\xi_w^2 \frac{\partial}{\partial \xi_w} \right) + \frac{1}{\xi_w^2} \frac{\partial}{\partial \mu} \left\{ \frac{2}{3} [1 - P_2(\mu)] \frac{\partial}{\partial \mu} \right\} ; \quad (21)$$

accordingly, Eq. (19) takes the explicit form:

$$\frac{1}{\xi_w^2} \frac{\partial}{\partial \xi_w} \left(\xi_w^2 \frac{\partial \theta_w}{\partial \xi_w} \right) + \frac{1}{\xi_w^2} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial \theta_w}{\partial \mu} \right] - v_w = - \sum (\Lambda_w \theta_w^{n_w}) ; \quad (22)$$

which shall be hereafter quoted as the EC2 equation. The related boundary conditions read [10], [5], [7]:

$$\theta_w(0, \mu) = 1 ; \quad \left(\frac{\partial \theta_w}{\partial \xi_w} \right)_{0, \mu} = 0 ; \quad \left(\frac{\partial \theta_w}{\partial \mu} \right)_{0, \mu} = 0 ; \quad (23)$$

$$\theta_w(\xi_w^{*-}, \mu) = \theta_w(\xi_w^{*+}, \mu) ; \quad (24)$$

$$\left(\frac{\partial \theta_w}{\partial \xi_w} \right)_{\xi_w^{*-}, \mu} = \left(\frac{\partial \theta_w}{\partial \xi_w} \right)_{\xi_w^{*+}, \mu} ; \quad \left(\frac{\partial \theta_w}{\partial \mu} \right)_{\xi_w^{*-}, \mu} = \left(\frac{\partial \theta_w}{\partial \mu} \right)_{\xi_w^{*+}, \mu} ; \quad (25)$$

where $\xi_w^{*-} < \xi_w^*$ and $\xi_w^{*+} > \xi_w^*$ are infinitely close to the interface, $\xi_w^* = \Xi_v, \xi_u^*$.

In terms of the parameters defined by Eq. (11), the EC2 equation, Eq. (22), via Eq. (12) translates into:

$$\frac{1}{\xi_{u1}^2} \frac{\partial}{\partial \xi_{u1}} \left(\xi_{u1}^2 \frac{\partial \theta_u}{\partial \xi_{u1}} \right) + \frac{1}{\xi_{u1}^2} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial \theta_u}{\partial \mu} \right] - v_{u1} = -\theta_u^{n_u} - \Lambda_{vu} \theta_v^{n_v} ; \quad (26)$$

$$\Lambda_{vu} = \frac{\Lambda_v}{\Lambda_u} = \frac{\lambda_v}{\lambda_u} ; \quad (27)$$

where the last term on the right-hand side of Eq. (26) could be interpreted as a tidal interaction due to the v subsystem.

Within the noncommon region, $\theta_v^{n_v}(\xi_v, \mu) = 0$, the EC2 equation reduces to its counterpart related to a single polytrope with polytropic index, n_u , central density, λ_u , central pressure, $\pi_u = K_u(\lambda_u^{1+1/n_u} - \rho_{b,u}^{1+1/n_u})$, scaling radius, α_{u1} , rotation parameter, v_{u1} , and related solutions formally coincide even if the boundary conditions are different. A special integral is $\theta_u^{(p)}(\xi_{u1}, \mu) = v_{u1}^{1/n_u}$.

In the limit of a vanishing inner (along the direction considered) subsystem, $\lambda_v \rightarrow 0$, $\Lambda_v \rightarrow 0$, $\Lambda_u \rightarrow 1$, $\alpha_{u1} \rightarrow \alpha_u$, $v_{u1} \rightarrow v_u$, $\xi_{u1} \rightarrow \xi_u$, and Eq. (26) reduces to its counterpart related to EC1 systems. On the other hand, $K_v \rightarrow +\infty$, $\Omega_v \rightarrow 0$, to ensure finite and nonzero α_v (implying the same for ξ_v), v_{v1} , respectively.

In the limit of a vanishing outer (along the direction considered) subsystem, $\lambda_u \rightarrow 0$, $\Lambda_u \rightarrow 0$, $\Lambda_v \rightarrow 1$, $\alpha_{v1} \rightarrow \alpha_v$, $v_{v1} \rightarrow v_v$, $\xi_{v1} \rightarrow \xi_v$, and Eq. (26) (where the indexes, u and v , are mutually exchanged) reduces to its counterpart related to EC1 systems. On the other hand, $K_u \rightarrow +\infty$, $\Omega_u \rightarrow 0$, to ensure finite and nonzero α_u (implying the same for ξ_u), v_{u1} , respectively. In addition, the gravitational potential within the noncommon region tends to the gravitational potential induced by the inner subsystem outside the interface, as:

$$4\pi G \lambda_v \alpha_u^2 \left\{ \theta_u(\xi_u, \mu) - \frac{1}{6} v_u \xi_u^2 [1 - P_2(\mu)] \right\} + \mathcal{V}_{b,u} = \mathcal{V}_v^{(\text{ext})}(\xi_v, \mu) + \mathcal{V}_{b,v} ; \quad (28)$$

where $\mathcal{V}_v^{(\text{ext})}$ can be expanded in Legendre polynomials e.g., [27], Chap. VII, §193.

If one subsystem is completely lying within the other, a vanishing outer component resembles generalized Roche models [24], Chap. IX, §234, keeping in mind the following differences: (i) the massive body is a polytrope instead of a homogeneous spheroid, and (ii) the vanishing atmosphere extends down to the centre instead of branching off from the boundary of the massive body.

The comparison between the alternative expressions of the gravitational potential, Eq. (18), yields:

$$\begin{aligned} & 4\pi G \sum (\lambda_w) \alpha_u^2 \left\{ \theta_u - \theta_{b,u} - \frac{1}{6} v_u \xi_u^2 [1 - P_2(\mu)] \right\} + \mathcal{V}_{b,u} \\ &= 4\pi G \sum (\lambda_w) \alpha_v^2 \left\{ \theta_v - \theta_{b,v} - \frac{1}{6} v_v \xi_v^2 [1 - P_2(\mu)] \right\} + \mathcal{V}_{b,v} ; \end{aligned} \quad (29)$$

which holds within and on the interface, $\xi_w \leq \xi_w^*$.

At the centre, $(\xi_w, \mu) = (0, \mu)$, Eq. (29) by use of (23) reduces to:

$$4\pi G \sum (\lambda_w) \alpha_u^2 (1 - \theta_{b,u}) + \mathcal{V}_{b,u} = 4\pi G \sum (\lambda_w) \alpha_v^2 (1 - \theta_{b,v}) + \mathcal{V}_{b,v} \quad ; \quad (30)$$

and the substitution of Eq. (30) into (29) after some algebra yields:

$$\begin{aligned} \alpha_u^2 \left\{ 1 - \theta_u + \frac{1}{6} v_u \xi_u^2 [1 - P_2(\mu)] \right\} &= \alpha_v^2 \left\{ 1 - \theta_v + \frac{1}{6} v_v \xi_v^2 [1 - P_2(\mu)] \right\} ; \\ \xi_w &\leq \xi_w^* \quad ; \end{aligned} \quad (31)$$

with no explicit boundary dependence.

The combination of Eqs. (13) and (31) after little algebra yields:

$$\begin{aligned} \theta_v(\xi_v, \mu) &= \theta_u(\xi_u, \mu) + (1 - \Gamma_{uv}) [1 - \theta_u(\xi_u, \mu)] \\ &\quad + \Gamma_{uv} \frac{v_v - v_u}{6} \xi_u^2 [1 - P_2(\mu)] \quad ; \quad \xi_w \leq \xi_w^* \quad ; \end{aligned} \quad (32)$$

$$\Gamma_{uv} = \frac{\alpha_u^2}{\alpha_v^2} \quad ; \quad (33)$$

where the third term on the right-hand side of Eq. (32) is null for subsystems rotating to the same extent ($v_u = v_v$) and the second is null for subsystems with coinciding scaling radii, $\Gamma_{uv} = 1$ or $\alpha_u = \alpha_v$.

Accordingly, four different situations can be considered concerning subsystems, namely:

- rotating to different extents and showing different scaling radii;
- rotating to different extents but showing equal scaling radii;
- rotating to the same extent but showing different scaling radii;
- rotating to the same extent and showing equal scaling radii.

With regard to the last case, the system reduces to a single matter distribution with central density, $\lambda = \lambda_u + \lambda_v$, and central pressure, $\pi = K_u(\lambda_u^{1+1/n_u} - \rho_{b,u}^{1+1/n_u}) + K_v(\lambda_v^{1+1/n_v} - \rho_{b,v}^{1+1/n_v})$. The additional restriction of equal polytropic indexes, $n_u = n_v = n$, implies a single reduced density profile, $\theta_u^{n_u} = \theta_v^{n_v} = \theta^n$ and hence $\rho = \rho_u + \rho_v = \lambda_u \theta_u^{n_u} + \lambda_v \theta_v^{n_v} = \lambda \theta^n$, where the central pressure reads $\pi = K(\lambda^{1+1/n} - \rho_b^{1+1/n})$, $K = K_u(\lambda_u^{1+1/n} - \rho_{b,u}^{1+1/n})/(\lambda^{1+1/n} - \rho_b^{1+1/n}) + K_v(\lambda_v^{1+1/n} - \rho_{b,v}^{1+1/n})/(\lambda^{1+1/n} - \rho_b^{1+1/n})$, conform to Dalton's law on gas partial pressure, and $\rho_b = \rho_{b,u} + \rho_{b,v}$ is the boundary density.

The particularization of Eq. (31) to the polar axis, $P_2(\mu) = 1$, shows that $\theta_u = 1 - \zeta$ implies $\theta_v = 1 - \Gamma_{uv} \zeta$ and, in particular, $\theta_v = 1 - \zeta = \theta_u$ for $\Gamma_{uv} = 1$

i.e. related isopycnic surfaces are tangent on the polar axis within the common region, $\xi_w \leq \xi_w^*$.

At the pole of the interface, $(\xi_i, \mu) = (\Xi_i, 1)$, $(\xi_j, \mu) = (\xi_j^*, 1)$, $\alpha_i \Xi_i = \alpha_j \xi_j^*$, $\theta_i = \theta_{b,i}$, $\theta_j = \theta_{b,j}^*$, and Eq. (29) reduces to:

$$\mathcal{V}_{b,i} = 4\pi G \sum (\lambda_w) \alpha_j^2 (\theta_{b,j}^* - \theta_{b,j}) + \mathcal{V}_{b,j} \quad ; \quad (34)$$

where $\Xi_i(\mu)$, $\xi_j^*(\mu)$, are the equations of the inner boundary in terms of related subsystems and $\theta_{b,j}^* = \theta_{b,j}(\xi_j^*, 1)$ denotes the isopycnic surface of the outer subsystem which is tangent to the interface on the pole, according to the above considerations.

The particularization of Eq. (32), $u = j$, $v = i$, to the pole of the interface, yields after some algebra:

$$\Gamma_{ji} = \frac{1 - \theta_{b,i}}{1 - \theta_{b,j}^*} \quad ; \quad (35)$$

where $\Gamma_{ji} \geq 1$ implies $\theta_{b,j}^* \geq \theta_{b,i}$ and vice versa.

The substitution of Eq. (35) into (31), by use of (33), after some algebra yields:

$$\alpha_i^2 \left\{ \theta_i - \theta_{b,i} - \frac{1}{6} v_i \xi_i^2 [1 - P_2(\mu)] \right\} = \alpha_j^2 \left\{ \theta_j - \theta_{b,j}^* - \frac{1}{6} v_j \xi_j^2 [1 - P_2(\mu)] \right\} \quad ; \quad (36)$$

which depends only on scaling radii, boundary densities, and rotation parameters.

On the polar axis, $P_2(\mu) = 1$, Eq. (36) reduces to:

$$\Gamma_{ji} = \frac{\theta_i - \theta_{b,i}}{\theta_j - \theta_{b,j}^*} \quad ; \quad (37)$$

which includes Eq. (35) and also holds, for any direction via Eq. (36), in the special case of subsystems rotating to the same extent, $v_i = v_j$, where the equipotential surface of the j subsystem, $\theta_j = \ell_j$, and the equipotential surface of the i subsystem, $\theta_i = \theta_{b,i} + \Gamma_{ji}(\ell_j - \theta_{b,j}^*) = \ell_i$, are coincident in the physical space, as expected.

Turning to the general case, equilibrium configurations may be determined to a selected order of approximation, using a method outlined in earlier attempts [10], [5], [7].

2.2 Collisionless fluids

The results of Subsection 2.1, related to multi-component (in particular, two-component) polytropes, can be extended from the collisional to the collisionless case following the same kind of procedure used for EC1 polytropes. To this

respect, it has been established an exact collisionless dynamical counterpart exists for collisional polytropes within the polytropic index range, $1/2 \leq n \leq 5$, [39], [21], Chap. 6, §6.1.9, [1], Chap. 4, §4.3.3. The same holds for the isothermal sphere [1], Chap. 4, §4.3.3. A similar method can be used in dealing with multi-component (in particular, two-component) polytropes. For detailed calculations, an interested reader is addressed to the above mentioned references.

Accordingly, the distribution function of the w subsystem reads:

$$f_w(x_1, x_2, x_3, v_1, v_2, v_3) = \begin{cases} C_w(H_w - H)^{n_w-3/2} & ; \quad H \leq H_w < 0 \quad ; \\ 0 & ; \quad H > H_w \quad ; \end{cases} \quad (38)$$

where C_w is a positive constant dimensioned as $[L^{-2n_w-3}T^{2n_w}]$ and H_w a negative constant dimensioned as $[L^2T^{-2}]$. In addition, the requirement of a finite mass implies $n_w > 1/2$. In the limiting case, $n_w \rightarrow 1/2$, Eq. (38) reduces to:

$$f_w(x_1, x_2, x_3, v_1, v_2, v_3) = C_w \delta(H_w - H) \quad ; \quad (39)$$

where $\delta(H_w - H)$ is the Dirac's delta dimensioned as $[L^{-2}T^2]$.

The integration of the distribution function, expressed by Eq. (38) and (39), over the coordinate space, yields the density distribution:

$$\rho_w(x_1, x_2, x_3) = \mu_w \psi_w^{n_w}(x_1, x_2, x_3) \quad ; \quad n_w \geq \frac{1}{2} \quad ; \quad (40)$$

$$\mu_w = 2^{5/2} \pi \overline{m}_w C_w B\left(\frac{3}{2}, n_w - \frac{1}{2}\right) \quad ; \quad (41)$$

$$\psi_w(x_1, x_2, x_3) = H_w + \mathcal{V}_G(x_1, x_2, x_3) + \frac{1}{2} \Omega_w^2 (x_1^2 + x_2^2) \quad ; \quad (42)$$

where \overline{m}_w is the mean particle mass of the w subsystem, \mathcal{V}_G is the total gravitational potential and $B(p, q)$ coincides with the Euler's complete beta function for $p > 0$, $q > 0$, and equals unity for $q = 0$ i.e. $n_w = 1/2$.

The comparison between Eqs. (2) and (40) shows that collisional and collisionless polytropic subsystems with equal density profiles and polytropic indexes, subjected to equal tidal and centrifugal potential, are related as:

$$\rho_w = \lambda_w \theta_w^{n_w} = \mu_w \psi_w^{n_w} \quad ; \quad (43)$$

which is equivalent to:

$$\psi_w = \left(\frac{\lambda_w}{\mu_w} \right)^{1/n_w} \theta_w \quad ; \quad (44)$$

where the ratio, $(\lambda_w/\mu_w)^{1/n_w}$, is dimensioned as a potential, $[L^2T^{-2}]$.

The Poisson equation is readily derived from Eq. (42) as:

$$\Delta \mathcal{V}_G + 2\Omega_w^2 = \Delta \psi_w = -4\pi G \rho + 2\Omega_w^2 \quad ; \quad (45)$$

where $\rho = \sum(\rho_w) = \sum(\mu_w \psi_w^{n_w})$ is the total density and $\psi_w = 0$ outside the w subsystem.

The mean square velocity of a collisionless fluid, $\overline{(v_w^2)}$, may be calculated using the theorem of the mean with regard to the phase space. In the case under discussion, the result is:

$$\overline{(v_w^2)} = \frac{3}{n_w + 1} \psi_w ; \quad n_w \geq \frac{1}{2} ; \quad (46)$$

which, under the restriction of isotropic stress tensor and nonrelativistic velocities, relates to the pressure as e.g., [21], Chap. 1, §1.7:

$$p_w = \frac{1}{3} \rho_w \overline{(v_w^2)} = \frac{1}{n_w + 1} \rho_w \psi_w ; \quad v_w \ll c ; \quad n_w \geq \frac{1}{2} ; \quad (47)$$

and the substitution of Eqs. (40) and (43) into (47) yields:

$$p_w = K_w \rho_w^{1+1/n_w} ; \quad (48)$$

$$K_w = \frac{1}{n_w + 1} \frac{1}{\mu_w^{1/n_w}} ; \quad n_w \geq \frac{1}{2} ; \quad (49)$$

a comparison between Eqs. (1) and (47) shows that the pressure of collisionless polytropic subsystems has the same formal expression of their dynamical collisional counterparts where the density on the boundary is null, $\rho_{b,w} = 0$, provided $n_w \geq 1/2$.

The pressure has necessarily to be null on the boundary, $p_{b,w} = 0$, which implies $\rho_{b,w} = 0$, $\psi_{b,w} = 0$. Accordingly, the particularization of Eq. (42) to the surface of the w subsystem yields:

$$H_w + \mathcal{V}_{b,w} = 0 ; \quad (50)$$

or $H_w = -\mathcal{V}_{b,w}$ i.e. the opposite of the potential induced by the whole system on the surface of the w subsystem.

The above results for collisionless polytropic subsystems may be formulated in terms of dimensionless variables and parameters, defined by Eq. (11), by use of Eq. (43), yielding the same formal expression of their dynamical collisional counterparts, provided $n_w \geq 1/2$.

The existence of collisionless polytropes with nonzero boundary densities, exhibiting equal configurations with respect to their collisional counterparts, remains (to the knowledge of the author) an open question.

2.3 The general problem

For axisymmetric configurations, the exact solutions of the EC2 equation, Eq. (22), may be expanded in series of even Legendre polynomials multiplied

by functions of the radial coordinate, $\theta_{2\ell,w}(\xi_w)$, and coefficients, $A_{2\ell,w}$, which depend on the rotation parameters provided $2\ell > 0$. Odd Legendre polynomials in the series expansion are ruled out by symmetry with respect to the equatorial plane. For further details, an interested reader is addressed to an earlier attempt [4].

2.3.1 The noncommon region

The presence of either subsystem within the noncommon region allows the notation be shortened for sake of simplicity, throughout the current Subsection, as:

$$\xi = \xi_{w1} \ ; \ v = v_{w1} \ ; \ n = n_w \ ; \ A_{2\ell} = A_{2\ell,w}^{(\text{ncm})} \ ; \ \theta_{2\ell} = \theta_{2\ell,w}^{(\text{ncm})} \ ; \quad (51)$$

where the apex, (ncm), denotes the noncommon region. Accordingly, the solution of Eq. (22) via (26) can be expanded in Legendre polynomials as [10], [7]:

$$\theta(\xi, \mu) = \sum_{\ell=0}^{+\infty} A_{2\ell} \theta_{2\ell}(\xi) P_{2\ell}(\mu) \ ; \quad (52)$$

$$P_\ell(\mu) = \frac{1}{2^\ell} \frac{1}{\ell!} \frac{d^\ell}{d\mu^\ell} [(\mu^2 - 1)^\ell] \ ; \quad |P_\ell(\mu)| \leq 1 \ ; \quad \ell = 0, 1, 2, \dots \ ; \quad (53)$$

$$\frac{d}{d\mu} \left[(1 - \mu^2) \frac{dP_\ell}{d\mu} \right] = -\ell(\ell + 1) P_\ell(\mu) \ ; \quad (54)$$

$$\lim_{v_w \rightarrow 0} A_{2\ell} = \delta_{2\ell,0} \ ; \quad (55)$$

where δ is the Kronekher symbol; $A_{2\ell}$, $2\ell > 0$, are coefficients which depend on the rotation parameters; $\theta_{2\ell}$ are the EC2 associated function of degree, 2ℓ ; and the Legendre polynomials, defined by Eq. (53), obey the Legendre equation, Eq. (54).

In absence of rotation, $v_u = v_v = 0$, the system attains the related undistorted spherical configuration and the EC2 associated function, $\theta_0(\xi)$, via Eq. (23) coincides with the EC2 function, $\theta(\xi, \mu)$, hence $A_0 = 1$ according to Eq. (55). Conversely, the coefficients, $A_{2\ell}$, $2\ell > 0$, depend on the rotation parameters and vanish for spherical configurations, according to Eq. (55).

If the distorsion due to rigid rotation may be considered as a small perturbation on the spherical shape, then the first term of the series expansion on the right-hand side of Eq. (52) is dominant, which implies the following inequality:

$$|R_1(\xi, \mu)| \ll |\theta_0(\xi)| \ ; \quad (56)$$

$$R_1(\xi, \mu) = \sum_{\ell=1}^{+\infty} A_{2\ell} \theta_{2\ell}(\xi) P_{2\ell}(\mu) \ ; \quad (57)$$

and the power on the right-hand side of Eq. (26) can safely be approximated as:

$$\theta^n(\xi, \mu) = \theta_0^n(\xi) + n\theta_0^{n-1}(\xi)R_1(\xi, \mu) ; \quad (58)$$

which is a series expansion in Legendre polynomials. With regard to the special cases, $n = 0, 1$, Eq. (58) is exact.

The series on the right-hand side of Eq. (52) describes the expansion of the nonrotating noncommon region as a whole, via θ_0 , and superimposed on this an oblateness, via R_1 . More specifically, θ_0 relates to an expanded spherical shell where the radial contribution of rigid rotation adds to the undistorted configuration, and R_1 quantifies the meridional distortion.

Let ξ_{ex} define the (fictitious) spherical isopycnic surface of the expanded spherical shell, as:

$$\theta_0(\xi_{\text{ex}}) = \theta(\xi_{\text{un}}) = \kappa ; \quad (59)$$

where the index, ex, un, denotes the expanded and the undistorted spherical shell, respectively, and radial expansion implies $\xi_{\text{un}} \leq \xi_{\text{ex}}$.

The substitution of Eqs. (57) and (59) into (52) yields:

$$R_1(\xi_{\text{ex}}, \mp \mu_{\text{ex}}) = 0 ; \quad (60)$$

and the locus, $(\xi_{\text{ex}}, \mp \mu_{\text{ex}})$, defines the intersection between the oblate isopycnic surface, $\theta(\xi, \mu) = \kappa$, and the (fictitious) spherical isopycnic surface, $\theta_0(\xi_{\text{ex}}) = \kappa$. In the nonrotating limit, $v \rightarrow 0$, $\xi_{\text{ex}} \rightarrow \xi_{\text{un}}$, and the term containing $P_2(\mu)$ is expected to be dominant with respect to the others in Eq. (57). Accordingly, μ_{ex} relates to $P_2(\mu) = 0$, hence $\mu_{\text{ex}} \rightarrow 1/\sqrt{3}$. For further details, an interested reader is addressed to the parent paper [9].

The substitution of Eqs. (52) and (58) into (26), after equating separately the terms of the same degree in Legendre polynomials, shows Eq. (26) is equivalent to the set of EC2 associated equations e.g., [6]:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta_0}{d\xi} \right) - v = -|\theta_0|^n \cos(n\pi\zeta) ; \quad (61)$$

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta_2}{d\xi} \right) - \frac{6}{\xi^2} \theta_2 = -n|\theta_0|^{n-1} \cos[(n-1)\pi\zeta] \theta_2 ; \quad (62)$$

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta_{2\ell}}{d\xi} \right) - \frac{(2\ell+1)2\ell}{\xi^2} \theta_{2\ell} = -n|\theta_0|^{n-1} \cos[(n-1)\pi\zeta] \theta_{2\ell} ; \quad (63)$$

$$\zeta = \begin{cases} 0 & ; \quad \theta_0 \geq 0 \\ 1 & ; \quad \theta_0 < 0 \end{cases} ; \quad (64)$$

where the occurrence of absolute values and cosines on the right-hand side of Eqs. (61)-(63), makes θ_0^n and θ_0^{n-1} , $\theta_0 < 0$, be evaluated as the real part of

the principal value of complex powers; the boundary conditions relate to the interface.

The solutions of the EC2 associated equations, Eqs. (61)-(63), may be expanded in Taylor series as:

$$\theta_{2\ell}(\xi) = \sum_{k=0}^{+\infty} a_{2\ell,k}(\xi_0)(\xi - \xi_0)^k ; \quad (65)$$

$$a_{2\ell,0}(\xi_0) = \theta_{2\ell}(\xi_0) ; \quad a_{2\ell,k}(\xi_0) = \frac{1}{k!} \left(\frac{d^k \theta_{2\ell}}{d\xi^k} \right)_{\xi_0} ; \quad (66)$$

where, in particular, $a_{2\ell,1}(\xi_0) = \theta'_{2\ell}(\xi_0)$ and $a_{2\ell,2}(\xi_0) = \theta''_{2\ell}(\xi_0)/2$. It can be seen the convergence radius tends to be null if the starting point, ξ_0 , tends to a singular point outside the origin, ξ_0^\dagger , $\theta_0(\xi_0^\dagger) = 0$, which implies $\theta_0(\xi)$ and $\theta_0(\xi_0)$ are both positive or negative. The first and the second derivative of the EC2 associated functions, $\theta'_{2\ell}(\xi)$ and $\theta''_{2\ell}(\xi)$, can readily be expanded in Taylor series via Eq. (65). For further details, an interested reader is addressed to the parent paper [6].

The powers of $|\theta_0|$ appearing on the right-hand side of Eqs. (61)-(63) can be expanded in series as e.g., [18]:

$$|\theta_0(\xi)|^x = |\theta_0(\xi_0)|^x \sum_{k=0}^{+\infty} C_k^{(x)}(\xi - \xi_0)^k ; \quad (67)$$

$$C_k^{(x)} = \frac{1}{\theta_0(\xi_0)} \frac{1}{k} \sum_{i=1}^k (ix - k + i) a_{0,i} C_{k-i}^{(x)} ; \quad C_0^{(x)} = 1 ; \quad (68)$$

where $a_{0,i}$ are defined by Eq. (66).

On the other hand, using Eqs. (65) and (66) yields:

$$|\theta_0(\xi)|^x = |\theta_0(\xi_0)|^x \left[1 + \frac{1}{\theta_0(\xi_0)} \sum_{k=1}^{+\infty} a_{0,k}(\xi - \xi_0)^k \right]^x ; \quad (69)$$

provided ξ_0 , ξ , both precede or exceed a singular point, ξ_0^\dagger , $\theta_0(\xi_0^\dagger) = 0$, i.e. $\xi_0 < \xi < \xi_0^\dagger$ or $\xi_0^\dagger < \xi_0 < \xi$. More specifically, $\theta_0(\xi)/\theta_0(\xi_0) > 0$ in the case under discussion, which implies the quantity within square brackets on the right-hand side of Eq. (69) is always positive.

The following identity:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta_{2\ell}}{d\xi} \right) = \theta''_{2\ell} + \frac{2}{\xi} \theta'_{2\ell} ; \quad (70)$$

implies use of the Taylor series expansion:

$$\frac{1}{\xi} = \frac{1}{\xi_0} \sum_{k=0}^{+\infty} (-1)^k \left(\frac{\xi - \xi_0}{\xi_0} \right)^k ; \quad |\xi - \xi_0| < \xi_0 ; \quad (71)$$

where the starting point, ξ_0 , has to be replaced by $\xi_0 + \Delta\xi < 2\xi_0$ whenever $\xi \geq 2\xi_0$, to ensure convergence.

The substitution of the series expansions for θ_0'' , θ_0' , $|\theta_0|^n$, $1/\xi$, via Eqs. (65)-(71), into (61), keeping in mind the coefficients of $(\xi - \xi_0)^k$ on both sides must necessarily be equal, yields after a lot of algebra [6]:

$$a_{0,k+2} = -\frac{1}{(k+1)(k+2)} \frac{1}{\xi_0} \left[\left(C_k^{(n)} \xi_0 + C_{k-1}^{(n)} \right) |\theta_0(\xi_0)|^n \cos(n\pi\zeta) + (k+1)(k+2)a_{0,k+1} \right] ; \quad k > 1 ; \quad (72a)$$

$$a_{0,0} = \theta_0(\xi_0) ; \quad a_{0,1} = \theta_0'(\xi_0) ; \quad (72b)$$

$$a_{0,2} = -\frac{1}{1 \cdot 2} \frac{1}{\xi_0} \left\{ C_0^{(n)} \xi_0 [|\theta_0(\xi_0)|^n \cos(n\pi\zeta) - v] + 1 \cdot 2 a_{0,1} \right\} ; \quad (72c)$$

$$a_{0,3} = -\frac{1}{2 \cdot 3} \frac{1}{\xi_0} \left[\left(C_1^{(n)} \xi_0 + C_0^{(n)} \right) |\theta_0(\xi_0)|^n \cos(n\pi\zeta) - v + 2 \cdot 3 a_{0,2} \right] ; \quad (72d)$$

which, together with Eqs. (67) and (68), make the series expansion, Eq. (65), $2\ell = 0$, be a solution of Eq. (61) for nonsingular starting points, ξ_0 , provided values of $\theta_0(\xi_0)$ and $\theta_0'(\xi_0)$ are known. For further details, an interested reader is addressed to the parent paper [6].

The factor, $6/\xi^2$, appearing on the left-hand side of Eq. (62), implies use of the Taylor series expansion:

$$\frac{1}{\xi^2} = \frac{1}{\xi_0^2} \sum_{k=0}^{+\infty} (-1)^k (k+1) \left(\frac{\xi - \xi_0}{\xi_0} \right)^k ; \quad |\xi - \xi_0| < \xi_0 ; \quad (73)$$

where the starting point, ξ_0 , has to be replaced by $\xi_0 + \Delta\xi < 2\xi_0$ whenever $\xi \geq 2\xi_0$, to ensure convergence.

The substitution of the series expansions for θ_2'' , θ_2' , θ_2 , $|\theta_0|^{n-1}$, $1/\xi$, $1/\xi^2$, via Eqs. (65)-(73), into (62), keeping in mind the coefficients of $(\xi - \xi_0)^k$ on both sides must necessarily be equal, yields after a lot of algebra [6]:

$$a_{2,k+2} = -\frac{a_{2,k+1}}{\xi_0} + \frac{1}{(k+1)(k+2)} \left\{ \frac{6}{\xi_0^2} \sum_{i=0}^k \frac{(-1)^i}{\xi_0^i} a_{2,k-i} - n |\theta_0(\xi_0)|^{n-1} \times \cos[(n-1)\pi\zeta] \left[\sum_{i=0}^k C_i^{(n-1)} a_{2,k-i} + \frac{1}{\xi_0} \sum_{i=0}^{k-1} C_i^{(n-1)} a_{2,k-i-1} \right] \right\} ; \quad k > 1 ; \quad (74a)$$

$$a_{2,0} = \theta_2(\xi_0) ; \quad a_{2,1} = \theta_2'(\xi_0) ; \quad (74b)$$

$$a_{2,2} = -\frac{a_{2,1}}{\xi_0} + \frac{1}{1 \cdot 2} \left\{ \frac{6}{\xi_0^2} a_{2,0} - n |\theta_0(\xi_0)|^{n-1} \cos[(n-1)\pi\zeta] a_{2,0} \right\} ; \quad (74c)$$

$$a_{2,3} = -\frac{a_{2,2}}{\xi_0} + \frac{1}{2 \cdot 3} \left\{ \frac{6}{\xi_0^2} \left[a_{2,1} - \frac{a_{2,0}}{\xi_0} \right] - n |\theta_0(\xi_0)|^{n-1} \cos[(n-1)\pi\zeta] \times \left[a_{2,1} + \left(C_1^{(n-1)} + \frac{C_0^{(n-1)}}{\xi_0} \right) a_{2,0} \right] \right\} ; \quad (74d)$$

which, together with Eqs. (67) and (68), make the series expansion, Eq. (65), $2\ell = 2$, be a solution of Eq. (62) for nonsingular starting points, ξ_0 , provided values of $\theta_2(\xi_0)$, $\theta'_2(\xi_0)$, $\theta_0(\xi_0)$ and $\theta'_0(\xi_0)$ are known. For further details, an interested reader is addressed to the parent paper [6].

The above procedure can be extended to the EC2 associated equations, Eqs. (63), $2\ell > 2$, to express related solutions as series expansions starting from nonsingular points. Nevertheless, the EC2 associated functions of order, $2\ell > 2$, are of little practical interest in that $A_{2\ell} = 0$ [10], [4] and for this reason they shall not be considered in the following.

In terms of the variable, ξ_w , the EC2 associated functions in the noncommon region, $\theta_{2\ell,w}(\xi_w)$, via Eqs. (11), (51), (65), can be expanded in Taylor series as:

$$\theta_{2\ell,w}(\xi_w) = \theta_{2\ell,w}(\xi_{0,w}) + \sum_{k=1}^{+\infty} a_{2\ell,k}^{(\text{ncm})}(\xi_{0,w})(\xi_w - \xi_{0,w})^k ; \quad \xi_w^* \leq \xi_w \leq \Xi_w ; \quad (75)$$

$$a_{2\ell,k}^{(\text{ncm})}(\xi_{0,w}) = \Lambda_w^{k/2} a_{2\ell,k}(\xi_{0,w1}) ; \quad (76)$$

where $n = n_w$, $v = v_{w1} = v_w/\Lambda_w$, $\xi = \xi_{w1} = \Lambda_w^{1/2}\xi_w$, when using Eqs. (72) and (74). The first starting point relates to the interface, $\xi_{0,w} = \xi_w^*$, which can be determined from the knowledge of the EC2 associated functions within the common region.

2.3.2 The common region

The presence of both subsystems within the common region allows the notation be shortened for sake of simplicity, throughout the current Subsection, as:

$$A_{2\ell,w} = A_{2\ell,w}^{(\text{com})} ; \quad \theta_{2\ell,w} = \theta_{2\ell,w}^{(\text{com})} ; \quad (77)$$

where the apex, (com), denotes the common region. Accordingly, the solution of Eq. (22) can be expanded in Legendre polynomials as:

$$\theta_w(\xi_w, \mu) = \sum_{\ell=0}^{+\infty} A_{2\ell,w} \theta_{2\ell,w}(\xi_w) P_{2\ell}(\mu) ; \quad (78)$$

$$\lim_{v_w \rightarrow 0} A_{2\ell,w} = \delta_{2\ell,0} ; \quad (79)$$

where δ is the Kronecker symbol; $A_{2\ell,w}$ are coefficients which depend on the rotation parameters; $\theta_{2\ell,w}$ are the EC2 associated function of degree, 2ℓ ; and the Legendre polynomials, defined by Eq. (53), obey the Legendre equation, Eq. (54).

In absence of rotation, $v_u = v_v = 0$, the system attains the related undistorted spherical configuration and the EC2 associated function, $\theta_{0,w}(\xi_w)$, via Eq. (23) coincides with the EC2 function, $\theta_w(\xi_w, \mu)$, hence $A_{0,w} = 1$ according

to Eq. (79). Conversely, the coefficients, $A_{2\ell,w}$, $2\ell > 0$, depend on the rotation parameters and vanish for spherical configurations, according to Eq. (79).

If the distortion due to rigid rotation may be considered as a small perturbation on the spherical shape, then the first term of the series expansion on the right-hand side of Eq. (78) is dominant, which implies the following inequality:

$$|R_{1,w}(\xi_w, \mu)| \ll |\theta_{0,w}(\xi_w)| \quad ; \quad (80)$$

$$R_{1,w}(\xi_w, \mu) = \sum_{\ell=1}^{+\infty} A_{2\ell,w} \theta_{2\ell,w}(\xi_w) P_{2\ell}(\mu) \quad ; \quad (81)$$

and the powers on the right-hand side of Eq. (22) can safely be approximated as:

$$\theta_w^{n_w}(\xi_w, \mu) = \theta_{0,w}^{n_w}(\xi_w) + n_w \theta_{0,w}^{n_w-1}(\xi_w) R_{1,w}(\xi_w, \mu) \quad ; \quad (82)$$

which is a series expansion in Legendre polynomials. With regard to the special cases, $n_w = 0, 1$, Eq. (82) is exact.

The series on the right-hand side of Eq. (78) describes the expansion of the nonrotating common region as a whole, via $\theta_{0,w}$, and superimposed on this an oblateness, via $R_{1,w}$. More specifically, $\theta_{0,w}$ relates to an expanded sphere where the radial contribution of rigid rotation adds to the undistorted configuration, and $R_{1,w}$ quantifies the meridional distortion.

Let $\xi_{\text{ex},w}$ define the (fictitious) spherical isopycnic surface of the expanded sphere, as:

$$\theta_{0,w}(\xi_{\text{ex},w}) = \theta(\xi_{\text{un},w}) = \kappa_w \quad ; \quad (83)$$

where the index, ex, un, denotes the expanded and the undistorted sphere, respectively, and radial expansion implies $\xi_{\text{un},w} \leq \xi_{\text{ex},w}$.

The substitution of Eqs. (81) and (83) into (78) yields:

$$R_{1,w}(\xi_{\text{ex},w}, \mp \mu_{\text{ex},w}) = 0 \quad ; \quad (84)$$

and the locus, $(\xi_{\text{ex},w}, \mp \mu_{\text{ex},w})$, defines the intersection between the oblate isopycnic surface, $\theta_w(\xi_w, \mu) = \kappa_w$, and the (fictitious) spherical isopycnic surface, $\theta_{0,w}(\xi_{\text{ex},w}) = \kappa_w$. In the nonrotating limit, $v_w \rightarrow 0$, $\xi_{\text{ex},w} \rightarrow \xi_{\text{un},w}$, and the term containing $P_2(\mu)$ is expected to be dominant with respect to the others in Eq. (81). Accordingly, $\mu_{\text{ex},w}$ relates to $P_2(\mu) = 0$, hence $\mu_{\text{ex},w} \rightarrow 1/\sqrt{3}$. For further details, an interested reader is addressed to the parent paper [9].

The substitution of Eqs. (78) and (82) into (22), after equating separately the terms of the same degree in Legendre polynomials, shows Eq. (22) is equivalent to the set of EC2 associated equations e.g., [6]:

$$\frac{1}{\xi_w^2} \frac{d}{d\xi_w} \left[\xi_w^2 \frac{d(A_{0,w} \theta_{0,w})}{d\xi_w} \right] - v_w = - \sum [\Lambda_w |A_{0,w} \theta_{0,w}|^{n_w} \cos(n_w \pi \zeta_w)] \quad ; \quad (85)$$

$$\begin{aligned} & \frac{1}{\xi_w^2} \frac{d}{d\xi_w} \left[\xi_w^2 \frac{d(A_{2,w}\theta_{2,w})}{d\xi_w} \right] - \frac{6}{\xi_w^2} A_{2,w}\theta_{2,w} \\ &= - \sum \left\{ \Lambda_w n_w |A_{0,w}\theta_{0,w}|^{n_w-1} \cos[(n_w-1)\pi\zeta_w] A_{2,w}\theta_{2,w} \right\} ; \end{aligned} \quad (86)$$

$$\begin{aligned} & \frac{1}{\xi_w^2} \frac{d}{d\xi_w} \left[\xi_w^2 \frac{d(A_{2\ell,w}\theta_{2\ell,w})}{d\xi_w} \right] - \frac{2\ell(2\ell+1)}{\xi_w^2} A_{2\ell,w}\theta_{2\ell,w} \\ &= - \sum \left\{ \Lambda_w n_w |A_{0,w}\theta_{0,w}|^{n_w-1} \cos[(n_w-1)\pi\zeta_w] A_{2\ell,w}\theta_{2\ell,w} \right\} ; \end{aligned} \quad (87)$$

$$\theta_{2\ell,w}(0) = \delta_{2\ell,0} ; \quad \theta'_{2\ell,w}(0) = 0 ; \quad (88)$$

$$\zeta_w = \begin{cases} 0 & ; \quad \theta_{0,w} \geq 0 ; \\ 1 & ; \quad \theta_{0,w} < 0 ; \end{cases} \quad (89)$$

where the occurrence of absolute values and cosines on the right-hand side of Eqs. (85)-(87), makes $\theta_{0,w}^{n_w}$ and $\theta_{0,w}^{n_w-1}$, $\theta_{0,w} < 0$, be evaluated as the real part of the principal value of complex powers; the boundary conditions relate to the centre.

The right-hand side of Eqs. (85)-(87) is independent of the subsystem under consideration, which implies $A_{2\ell,u} = A_{2\ell,v} = A_{2\ell}$, as shown in Appendix A. Accordingly, Eqs. (85)-(87) reduce to:

$$\frac{1}{\xi_w^2} \frac{d}{d\xi_w} \left(\xi_w^2 \frac{d\theta_{0,w}}{d\xi_w} \right) - v_w = - \sum [\Lambda_w |\theta_{0,w}|^{n_w} \cos(n_w \pi \zeta_w)] ; \quad (90)$$

$$\begin{aligned} & \frac{1}{\xi_w^2} \frac{d}{d\xi_w} \left(\xi_w^2 \frac{d\theta_{2,w}}{d\xi_w} \right) - \frac{6}{\xi_w^2} \theta_{2,w} \\ &= - \sum \left\{ \Lambda_w n_w |\theta_{0,w}|^{n_w-1} \cos[(n_w-1)\pi\zeta_w] \theta_{2,w} \right\} ; \end{aligned} \quad (91)$$

$$\begin{aligned} & \frac{1}{\xi_w^2} \frac{d}{d\xi_w} \left(\xi_w^2 \frac{d\theta_{2\ell,w}}{d\xi_w} \right) - \frac{2\ell(2\ell+1)}{\xi_w^2} \theta_{2\ell,w} \\ &= - \sum \left\{ \Lambda_w n_w |\theta_{0,w}|^{n_w-1} \cos[(n_w-1)\pi\zeta_w] \theta_{2\ell,w} \right\} ; \end{aligned} \quad (92)$$

where the explicit dependence on $A_{2\ell,w}$ has been erased.

Related solutions of the EC2 associated equations, Eqs. (90)-(92), may be expanded in Taylor series as:

$$\theta_{2\ell,w}(\xi_w) = \sum_{k=0}^{+\infty} a_{2\ell,k}^{(w,w)}(\xi_{0,w})(\xi_w - \xi_{0,w})^k ; \quad (93)$$

$$a_{2\ell,0}^{(w,w)}(\xi_{0,w}) = \theta_{2\ell,w}(\xi_{0,w}); \quad a_{2\ell,k}^{(w,w)}(\xi_{0,w}) = \frac{1}{k!} \left(\frac{d^k \theta_{2\ell,w}}{d\xi_w^k} \right)_{\xi_{0,w}} ; \quad (94)$$

where, in particular, $a_{2\ell,1}^{(w,w)}(\xi_{0,w}) = \theta'_{2\ell,w}(\xi_{0,w})$ and $a_{2\ell,2}^{(w,w)}(\xi_{0,w}) = \theta''_{2\ell,w}(\xi_{0,w})/2$. It can be seen the convergence radius tends to be null if the starting point, $\xi_{0,w}$, tends to a singular point outside the origin, $\xi_{0,w}^\dagger$, $\theta_{0,w}(\xi_{0,w}^\dagger) = 0$, which implies

$\theta_{0,w}(\xi_w)$ and $\theta_{0,w}(\xi_{0,w})$ are both positive or negative. The first and the second derivative of the EC2 associated functions, $\theta'_{2\ell,w}(\xi_w)$ and $\theta''_{2\ell,w}(\xi_w)$, can readily be expanded in Taylor series via Eq. (93). For further details, an interested reader is addressed to the parent paper [6].

The solutions of the EC2 associated equations related to the other subsystem, let it be v , $w = u$, along the direction considered, may be expanded in Taylor series as shown in Appendix A.

The powers of $|\theta_{0,w}|$ appearing on the right-hand side of Eqs. (90)-(92) can be expanded in series as e.g., [18]:

$$|\theta_{0,w}(\xi_w)|^x = |\theta_{0,w}(\xi_{0,w})|^x \sum_{k=0}^{+\infty} C_{k,w}^{(x)} (\xi_w - \xi_{0,w})^k ; \quad (95)$$

$$C_{k,w}^{(x)} = \frac{1}{\theta_{0,w}(\xi_{0,w})} \frac{1}{k} \sum_{i=1}^k (ix - k + i) a_{0,i}^{(w,u)} C_{k-i,w}^{(x)} ; \quad C_{0,w}^{(x)} = 1 ; \quad (96)$$

where $w = u, v$ and $a_{0,i}^{(w,u)}$ are defined by Eqs. (94) and (330) written in Appendix A.

On the other hand, the combination of Eqs. (93) and (94) yields:

$$|\theta_{0,w}(\xi_u)|^x = |\theta_{0,w}(\xi_{0,u})|^x \left[1 + \frac{1}{\theta_{0,w}(\xi_{0,u})} \sum_{k=1}^{+\infty} a_{0,k}^{(w,u)} (\xi_u - \xi_{0,u})^k \right]^x ; \quad (97)$$

provided $\xi_{0,u}$, ξ_u , both precede or exceed a singular point, $\xi_{0,u}^\dagger$, $\theta_{0,u}(\xi_{0,u}^\dagger) = 0$, i.e. $\xi_{0,u} < \xi_u < \xi_{0,u}^\dagger$ or $\xi_{0,u}^\dagger < \xi_{0,u} < \xi_u$. More specifically, $\theta_{0,w}(\xi_u)/\theta_{0,w}(\xi_{0,u}) > 0$ in the case under discussion, which implies the quantity within square brackets on the right-hand side of Eq. (97) is always positive.

The following identity:

$$\frac{1}{\xi_w^2} \frac{d}{d\xi_w} \left(\xi_w^2 \frac{d\theta_{2\ell,w}}{d\xi} \right) = \theta''_{2\ell,w} + \frac{2}{\xi_w} \theta'_{2\ell,w} ; \quad (98)$$

implies use of the Taylor series expansion:

$$\frac{1}{\xi_w} = \frac{1}{\xi_{0,w}} \sum_{k=0}^{+\infty} (-1)^k \left(\frac{\xi_w - \xi_{0,w}}{\xi_{0,w}} \right)^k ; \quad |\xi_w - \xi_{0,w}| < \xi_{0,w} ; \quad (99)$$

where the starting point, $\xi_{0,w}$, has to be replaced by $\xi_{0,w} + \Delta\xi_w < 2\xi_{0,w}$ whenever $\xi_w \geq 2\xi_{0,w}$, to ensure convergence.

The substitution of the series expansions for $\theta''_{0,w}$, $\theta'_{0,w}$, $|\theta_{0,w}|^{n_w}$, $1/\xi_w$, via Eqs. (93)-(99), into (90), keeping in mind the coefficients of $(\xi_u - \xi_{0,u})^k$ on both sides must necessarily be equal, yields after a lot of algebra [6]:

$$a_{0,k+2}^{(u,u)} = -\frac{1}{(k+1)(k+2)} \frac{1}{\xi_{0,u}} \left\{ \sum \left\{ \left[C_{k,w}^{(n_w)} \xi_{0,u} + C_{k-1,w}^{(n_w)} \right] \Lambda_w |\theta_{0,w}(\xi_{0,u})|^{n_w} \right. \right.$$

$$\times \cos(n_w \pi \zeta_w) \} + (k+1)(k+2) a_{0,k+1}^{(u,u)} \} \quad ; \quad k > 1 \quad ; \quad (100a)$$

$$a_{0,0}^{(u,u)} = \theta_{0,u}(\xi_{0,u}) \quad ; \quad a_{0,1}^{(u,u)} = \theta'_{0,u}(\xi_{0,u}) \quad ; \quad (100b)$$

$$a_{0,2}^{(u,u)} = -\frac{1}{1 \cdot 2} \frac{1}{\xi_{0,u}} \left\{ \sum \left[C_{0,w}^{(n_w)} \xi_{0,u} \Lambda_w |\theta_{0,w}(\xi_{0,u})|^{n_w} \cos(n_w \pi \zeta_w) \right] \right. \\ \left. - C_{0,u}^{(n_u)} \xi_{0,u} v_u + 1 \cdot 2 a_{0,1}^{(u,u)} \right\} ; \quad (100c)$$

$$a_{0,3}^{(u,u)} = -\frac{1}{2 \cdot 3} \frac{1}{\xi_{0,u}} \left\{ \sum \left\{ \left[C_{1,w}^{(n_w)} \xi_{0,u} + C_{0,w}^{(n_w)} \right] \Lambda_w |\theta_{0,w}(\xi_{0,u})|^{n_w} \cos(n_w \pi \zeta_w) \right\} \right. \\ \left. - v_u + 2 \cdot 3 a_{0,2}^{(u,u)} \right\} ; \quad (100d)$$

which, together with Eqs. (95) and (96), make the series expansion, Eq. (93), $2\ell = 0$, be a solution of Eq. (90) for nonsingular starting points, $\xi_{0,u}$, provided values of $\theta_{0,u}(\xi_{0,u})$ and $\theta'_{0,u}(\xi_{0,u})$ are known. For further details, an interested reader is addressed to the parent paper [6]. The coefficients, $a_{0,k+2}^{(v,u)}$, needed for the calculation of $\theta_{0,v}(\xi_u)$ via Eq. (329), are related to their counterparts, $a_{0,k+2}^{(u,u)}$, via Eq. (333), as shown in Appendix A.

The series, defined by Eq. (93), where the coefficients are expressed by Eq. (100), represents a solution of the associated EC2 equation of degree 0 for nonsingular starting points, $\xi_{0,u}$, with regard to assigned input parameters and boundary conditions. For the central singular starting point, $\xi_{0,u} = 0$, Eq. (93) reduces to the MacLaurin series expansion:

$$\theta_{2\ell,w}(\xi_w) = \sum_{k=0}^{+\infty} a_{2\ell,k}^{(w,w)}(0) \xi_w^k \quad ; \quad (101)$$

$$a_{2\ell,0}^{(w,w)}(0) = \theta_{2\ell,w}(0) = \delta_{2\ell,0}; \quad a_{2\ell,k}^{(w,w)}(0) = \frac{1}{k!} \left(\frac{d^k \theta_{2\ell,w}}{d\xi_w^k} \right)_0 \quad ; \quad (102)$$

which implies the series expansion of the ratio, $\theta'_{2\ell,w}(\xi_w)/\xi_w$, as:

$$\frac{\theta'_{2\ell,w}(\xi_w)}{\xi_w} = \sum_{k=0}^{+\infty} k a_{2\ell,k}^{(w,w)}(0) \xi_w^{k-2} \quad ; \quad (103)$$

where $a_{2\ell,1}^{(w,w)}(0) = 0$ according to Eqs. (88) and (103). In addition, Eqs. (95) and (96) reduce to:

$$|\theta_{0,w}(\xi_w)|^x = 1 + \sum_{k=1}^{+\infty} C_{k,w}^{(x)} \xi_w^k \quad ; \quad (104)$$

$$C_{k,w}^{(x)} = \frac{1}{k} \sum_{i=1}^k (ix - k + i) a_{0,i}^{(w,u)} C_{k-i,w}^{(x)} \quad ; \quad C_{0,w}^{(x)} = 1 \quad ; \quad (105)$$

where $w = u, v$ and $a_{0,i}^{(w,u)}$ are defined by Eqs. (102) and (330) in Appendix A, where $\xi_{0,u} = 0$.

In the case under discussion, $2\ell = 0$, following the procedure used for nonsingular starting points yields:

$$a_{0,2k+2}^{(u,u)} = -\frac{1}{2k(2k+2)(2k+3)} \times \sum \left\{ \Lambda_w \sum_{i=1}^k \frac{(2in_w - 2k + 2i)(2k - 2i + 2)(2k - 2i + 3)}{\Gamma_{uw}(1 - \delta_{ik}v_w)} a_{0,2i}^{(w,u)} a_{0,2k-2i+2}^{(w,u)} \right\};$$

$$k > 0; \quad (106a)$$

$$a_{0,0}^{(u,u)} = 1; \quad a_{0,1}^{(u,u)} = 0; \quad a_{0,2}^{(u,u)} = -\frac{1 - v_u}{6}; \quad (106b)$$

$$a_{0,2k+1}^{(u,u)} = 0; \quad C_{2k+1,u}^{(n_u)} = 0; \quad k > 0; \quad C_{1,u}^{(n_u)} = a_{0,1}^{(u,u)} = 0; \quad (106c)$$

where the coefficients, $a_{0,2i}^{(v,u)}$, needed for the calculation of $a_{0,2k+2}^{(u,u)}$, are related to their counterparts, $a_{0,2i}^{(u,u)}$, via Eq. (334) as shown in Appendix A.

The series, defined by Eq. (101), where the coefficients are expressed by Eq. (106), represents a solution of the associated EC2 equation of degree, 0, for the singular starting point, $\xi_{0,u} = 0$, and the boundary conditions, $\theta_{0,u}(0) = 1$, $\theta'_{0,u}(0) = 0$.

The factor, $6/\xi_w^2$, appearing on the left-hand side of Eq. (91), implies use of the Taylor series expansion:

$$\frac{1}{\xi_w^2} = \frac{1}{\xi_{0,w}^2} \sum_{k=0}^{+\infty} (-1)^k (k+1) \left(\frac{\xi_w - \xi_{0,w}}{\xi_{0,w}} \right)^k; \quad |\xi_w - \xi_{0,w}| < \xi_{0,w}; \quad (107)$$

where the starting point, $\xi_{0,w}$, has to be replaced by $\xi_{0,w} + \Delta\xi_w < 2\xi_{0,w}$ whenever $\xi_w \geq 2\xi_{0,w}$, to ensure convergence.

The substitution of the series expansions for $\theta''_{2,w}$, $\theta'_{2,w}$, $\theta_{2,w}$, $|\theta_{0,w}|^{n_w-1}$, $1/\xi_w$, $1/\xi_w^2$, by use of Eqs. (93)-(99), (107), into (91), keeping in mind the coefficients of $(\xi_w - \xi_{0,w})^k$ on both sides must necessarily be equal, yields after a lot of algebra [6]:

$$a_{2,k+2}^{(u,u)} = -\frac{a_{2,k+1}^{(u,u)}}{\xi_{0,u}} + \frac{1}{(k+1)(k+2)} \left\{ \frac{6}{\xi_{0,u}^2} \sum_{i=0}^k \frac{(-1)^i}{\xi_{0,u}^i} a_{2,k-i}^{(u,u)} \right.$$

$$- \sum \left\{ \Lambda_w n_w |\theta_{0,w}(\xi_{0,w})|^{n_w-1} \cos[(n_w - 1)\pi\zeta_w] \right.$$

$$\times \left[\sum_{i=0}^k C_{i,w}^{(n_w-1)} a_{2,k-i}^{(w,u)} + \frac{1}{\xi_{0,u}} \sum_{i=0}^{k-1} C_{i,w}^{(n_w-1)} a_{2,k-i-1}^{(w,u)} \right] \left. \right\}; \quad k > 1; \quad (108a)$$

$$a_{2,0}^{(u,u)} = \theta_{2,u}(\xi_{0,u}); \quad a_{2,1}^{(u,u)} = \theta'_{2,u}(\xi_{0,u}); \quad (108b)$$

$$a_{2,2}^{(u,u)} = -\frac{a_{2,1}^{(u,u)}}{\xi_{0,u}} + \frac{1}{1 \cdot 2} \left\{ \frac{6}{\xi_{0,u}^2} a_{2,0}^{(u,u)} - \sum \left\{ \Lambda_w n_w |\theta_{0,w}(\xi_{0,w})|^{n_w-1} \right. \right. \\ \left. \left. \times \cos[(n_w - 1)\pi\zeta_w] C_{0,w}^{(n_w-1)} a_{2,0}^{(w,u)} \right\} \right\} ; \quad (108c)$$

$$a_{2,3}^{(u,u)} = -\frac{a_{2,2}^{(u,u)}}{\xi_{0,u}} + \frac{1}{2 \cdot 3} \left\{ \frac{6}{\xi_{0,u}^2} \left[a_{2,1}^{(u,u)} - \frac{a_{2,0}^{(u,u)}}{\xi_{0,u}} \right] - \sum \left\{ \Lambda_w n_w |\theta_{0,w}(\xi_{0,w})|^{n_w-1} \right. \right. \\ \left. \left. \times \cos[(n_w - 1)\pi\zeta_w] \left[a_{2,1}^{(w,u)} + \left(C_{1,w}^{(n_w-1)} + \frac{C_{0,w}^{(n_w-1)}}{\xi_{0,w}} \right) a_{2,0}^{(w,u)} \right] \right\} \right\} ; \quad (108d)$$

which, together with Eqs. (95) and (96), make the series expansion, Eq. (93), $2\ell = 2$, be a solution of Eq. (91) for nonsingular starting points, $\xi_{0,u}$, provided values of $\theta_{2,w}(\xi_{0,u})$, $\theta'_{2,w}(\xi_{0,u})$, $\theta_{0,w}(\xi_{0,w})$ and $\theta'_{0,w}(\xi_{0,w})$ are known. For further details, an interested reader is addressed to the parent paper [6]. The coefficients, $a_{2,k+2}^{(v,u)}$, needed for the calculation of $\theta_{2,v}(\xi_{0,v})$ via Eq. (329), are related to their counterparts, $a_{2,k+2}^{(u,u)}$, via Eq. (333), as shown in Appendix A.

For the central singular starting point, $\xi_{0,u} = 0$, following the procedure used for nonsingular starting points yields:

$$a_{2,k+2}^{(u,u)} = -\frac{1}{(2k+2)(2k+3)-6} \sum \left\{ \Lambda_w n_w \sum_{i=0}^k C_{i,w}^{(n_w-1)} a_{2,2k-2i}^{(w,u)} \right\} ; k > 1; \quad (109a)$$

$$a_{2,0}^{(u,u)} = 0 ; \quad a_{2,1}^{(u,u)} = 0 ; \quad (109b)$$

$$a_{2,2}^{(u,u)} (2 \cdot 3 - 6) = - \sum \left\{ \Lambda_w n_w C_{0,w}^{(n_w-1)} a_{2,0}^{(w,u)} \right\} ; \quad (109c)$$

$$a_{2,2k+1}^{(u,u)} = 0 ; \quad C_{2k+1,w}^{(n_w-1)} = 0 ; \quad (109d)$$

where the coefficients, $a_{2,2k-2i}^{(v,u)}(\xi_{0,v})$, needed for the calculation of $a_{2,2k+2}^{(u,u)}(\xi_{0,u})$, are related to their counterparts, $a_{2,2k-2i}^{(u,u)}(\xi_{0,u})$, via Eq. (334) as shown in Appendix A.

The coefficient, $a_{2,2}^{(u,u)}(0)$, is left undetermined according to Eq. (109c). On the other hand, the EC2 associated function, $\theta_{2,w}(\xi_w)$, remains undefined by a constant factor, A_2 , according to Eq. (91). Without loss of generality, $\theta_{2,w}(\xi_w)$ may be fixed as:

$$a_{2,2}^{(u,u)}(0) = 1 ; \quad (110)$$

where A_2 has still to be determined. The series, defined by Eq. (101), $2\ell = 2$, when the coefficients are expressed by Eqs. (109)-(110), represents a solution of the EC2 associated equation of degree, $2\ell = 2$, for the singular starting point, $\xi_{0,u} = 0$, and the boundary conditions, $\theta_{2,w}(0) = 0$, $\theta'_{2,w}(0) = 0$.

2.4 Behaviour of EC2 associated functions near singularities outside the origin

Owing to uniform convergence of the power series, defined by Eqs. (75), (93), differentiation or integration can be performed term by term within the convergence radius, $|\xi_w - \xi_{0,w}| < \Delta_C \xi_w$. Unfortunately, related coefficients are expressed in a rather cumbersome form via the recursion formulae, Eqs. (72), (74), (100), (106), (108), (109), which implies no simple means for analysing series convergence, leaving aside a few special cases. It can be seen a lower limit to the convergence radius satisfies $\Delta_C \xi_w > 1$ for the singular starting point, $\xi_{0,w} = 0$, and an upper limit satisfies $\Delta_C \xi_w < \min(\xi_{0,w}, |\xi_{0,w}^\dagger - \xi_{0,w}|)$, where $\xi_{0,w}^\dagger$ is the zero of $\theta_{0,w}$ which is nearest to $\xi_{0,w}$. For further details, an interested reader is addressed to the parent papers [36], [29], [6].

The power series, expressed by Eqs. (75), (93), do not converge as $\xi_{0,w} \rightarrow \xi_{0,w}^\dagger$, [6] which implies the convergence radius tends to be null as the starting point approaches a zero of $\theta_{0,w}$, $\lim_{\xi_{0,w} \rightarrow \xi_{0,w}^\dagger} \Delta_C \xi_w = 0$. Accordingly, the starting point, $\xi_{0,w}$, and the ending point on the convergence radius, $\xi_{0,w} + \Delta_C \xi_w$, lie on the same side with respect to the nearest singular point, $\xi_{0,w}^\dagger$. More specifically, $|\xi_w - \xi_{0,w}| < \Delta_C \xi_w$, $\xi_w < \xi_{0,w}^\dagger$, and an approximation is needed for evaluating the EC2 associated function on points, $\xi_w > \xi_{0,w}^\dagger$, via series expansions expressed by Eqs. (75), (93).

With regard to the EC2 equation, Eq. (22), related to the common region, it may safely be assumed the term containing $|\theta_v(\xi_v, \mu)|^{n_v}$, where v relates to the inner boundary along the direction considered, is negligible in comparison with the other ones in the neighbourhood of a singular point, $|\xi_{0,v}^\dagger - \xi_{0,v}| < \epsilon_v$, where ϵ_v is an assigned tolerance. Accordingly, the EC2 associated equations, Eqs. (90)-(92), $w = u$, reduce to Eqs. (61)-(63), respectively, via Eqs. (11) and (51). Then power-series solutions of the EC2 associated equations are expressed by Eq. (65) and related coefficients, for $2\ell = 0, 2$, by Eqs. (72), (74), particularized to the case under discussion, including the singular starting point, $\xi_{0,w} = 0$. Finally, the EC2 associated functions, $w = v$, can be determined within the common region, via Eq. (328), and related coefficients appearing in series expansions via Eqs. (333)-(334), as shown in Appendix A.

With regard to the EC2 equation, Eq. (22), related to the noncommon region, it may safely be assumed the term containing $|\theta_u(\xi_u, \mu)|^{n_u}$, where u relates to the outer boundary along the direction considered, is negligible in comparison with the other ones in the neighbourhood of a singular point, $|\xi_{0,u}^\dagger - \xi_{0,u}| < \epsilon_u$, where ϵ_u is an assigned tolerance. Accordingly, the EC2 equation, Eq. (22), reduces to:

$$\frac{1}{\xi_u^2} \frac{\partial}{\partial \xi_u} \left(\xi_u^2 \frac{\partial \theta_u}{\partial \xi_u} \right) + \frac{1}{\xi_u^2} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial \theta_u}{\partial \mu} \right] - v_u = 0 ; \quad (111)$$

and it can be verified via Eq. (54) a solution is:

$$\theta_u(\xi_u, \mu) = \sum_{\ell=0}^{+\infty} \left[D_{2\ell} \xi_u^{2\ell} + \frac{\overline{D}_{2\ell}}{\xi_u^{2\ell+1}} \right] P_{2\ell}(\mu) + \frac{1}{6} v_u \xi_u^2 [1 - P_2(\mu)] \quad ; \quad (112)$$

where $D_{2\ell}$, $\overline{D}_{2\ell}$, are constants to be determined.

With regard to the first nonzero singular point, $\xi_{0,u}^\dagger = \Xi_{\text{ex},u}$, the comparison between Eqs. (52) and (112) at a specified transition point, $\xi_u = \hat{\xi}_u < \Xi_{\text{ex},u}$, equating the terms of equal degree in Legendre polynomials, yields:

$$A_0 \theta_{0,u}(\hat{\xi}_u) = D_0 + \frac{\overline{D}_0}{\hat{\xi}_u} + \frac{1}{6} v_u \hat{\xi}_u^2 \quad ; \quad A_0 = 1 \quad ; \quad (113)$$

$$A_2 \theta_{2,u}(\hat{\xi}_u) = D_2 \hat{\xi}_u^2 + \frac{\overline{D}_2}{\hat{\xi}_u^3} - \frac{1}{6} v_u \hat{\xi}_u^2 \quad ; \quad (114)$$

$$A_{2\ell} \theta_{2\ell,u}(\hat{\xi}_u) = D_{2\ell} \hat{\xi}_u^{2\ell} + \frac{\overline{D}_{2\ell}}{\hat{\xi}_u^{2\ell+1}} \quad ; \quad 2\ell > 2 \quad ; \quad (115)$$

where the constants, D_0 , \overline{D}_0 ; D_2 , \overline{D}_2 ; $D_{2\ell}$, $\overline{D}_{2\ell}$; can be inferred from the continuity of the gravitational potential and the radial component of the gravitational force, which implies the continuity of the EC2 associated functions and their first derivatives on the transition point, $\hat{\xi}_u$.

To this respect, it suffices solving the system of Eqs. (113), (114), (115), and related radial first derivatives on both sides:

$$A_0 \theta'_{0,u}(\hat{\xi}_u) = -\frac{\overline{D}_0}{\hat{\xi}_u^2} + \frac{1}{3} v_u \hat{\xi}_u \quad ; \quad A_0 = 1 \quad ; \quad (116)$$

$$A_2 \theta'_{2,u}(\hat{\xi}_u) = 2D_2 \hat{\xi}_u - \frac{3\overline{D}_2}{\hat{\xi}_u^4} - \frac{1}{3} v_u \hat{\xi}_u \quad ; \quad (117)$$

$$A_{2\ell} \theta'_{2\ell,u}(\hat{\xi}_u) = 2\ell D_{2\ell} \hat{\xi}_u^{2\ell-1} - \frac{(2\ell+1)\overline{D}_{2\ell}}{\hat{\xi}_u^{2\ell+2}} \quad ; \quad 2\ell > 2 \quad ; \quad (118)$$

respectively. After some algebra, the result is:

$$D_{2\ell} = \frac{1}{4\ell+1} \frac{1}{\hat{\xi}_u^{2\ell}} \times \left\{ A_{2\ell} [(2\ell+1)\theta_{2\ell,u}(\hat{\xi}_u) + \hat{\xi}_u \theta'_{2\ell,u}(\hat{\xi}_u)] - (2\ell+3) \frac{\delta_{2\ell,0} - \delta_{2\ell,2}}{6} v_u \hat{\xi}_u^2 \right\} ; \quad (119)$$

$$\overline{D}_{2\ell} = \frac{\hat{\xi}_u^{2\ell+1}}{4\ell+1} \times \left\{ A_{2\ell} [2\ell\theta_{2\ell,u}(\hat{\xi}_u) - \hat{\xi}_u \theta'_{2\ell,u}(\hat{\xi}_u)] - (2\ell-2) \frac{\delta_{2\ell,0} - \delta_{2\ell,2}}{6} v_u \hat{\xi}_u^2 \right\} ; \quad (120)$$

where the transition point may be related to the first nonzero singular point, as $\hat{\xi}_u = \zeta_u \Xi_{\text{ex},u}$, $0 < 1 - \zeta_u \ll 1$.

It is worth noticing the counterpart of Eq. (114) in the parent paper [6], Eq. (24) therein, lacks the last quadratic term for the following reason. The EC2 associated function, $\theta_{2,u}$, has been inferred from a solution of the EC2 equation in the current paper and from a solution of the EC2 associated equation of degree, $2\ell = 2$, in the parent paper, where the former expression is exact and the latter is approximate, in the case under discussion.

On the basis of the above results, the behaviour of the EC2 associated functions, $\theta_{0,w}$, $\theta_{2,w}$, in the neighbourhood of first nonzero singular point, $\xi_{0,w}^\dagger = \Xi_{\text{ex},w}$, can be ascertained via Eqs. (65), (72), (74), for the inner boundary, $w = v$ along the direction considered, and Eqs. (113)-(120) for the outer boundary, $w = u$ along the direction considered.

Similar results can be inferred for other singular points via Eqs. (93), (100), (108), outside the boundary in connection with the analytical continuation of $\theta_{2\ell,u}(\xi_u)$.

2.5 Special cases

For a restricted number of special cases, the solution of both EC2 and associated EC2 equations may be expressed analytically. To save space and simplify the notation, the common and noncommon region shall be dealt with in different Subsubsections. Accordingly, it shall be intended that unlabelled symbols in each Subsubsection are related to the region under discussion therein.

2.5.1 The noncommon region

With regard to the noncommon region, both the EC2 equation, Eq. (26), and the EC2 associated equations, Eqs. (61)-(63), formally coincide with their counterparts related to EC1 polytropes. On the other hand, the boundary conditions are related to the interface instead of the origin, which implies general solutions with doubled integration constants.

Concerning the special case, $(n_v, n_u) = (n_v, 0)$, the EC2 equation, Eq. (22), reduces to:

$$\frac{1}{\xi_u^2} \frac{\partial}{\partial \xi_u} \left(\xi_u^2 \frac{\partial \theta_u}{\partial \xi_u} \right) + \frac{1}{\xi_u^2} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial \theta_u}{\partial \mu} \right] - v_u = -\Lambda_u ; \quad (121)$$

which can be turned into Eq. (111) provided the constant, v_u , is replaced therein by $v_u - \Lambda_u$, which preserves related results. In particular, a solution of Eq. (121) is:

$$\theta_u(\xi_u, \mu) = \sum_{\ell=0}^{+\infty} \left[D_{2\ell} \xi_u^{2\ell} + \frac{\overline{D}_{2\ell}}{\xi_u^{2\ell+1}} \right] P_{2\ell}(\mu) + \frac{1}{6} (v_u - \Lambda_u) \xi_u^2 [1 - P_2(\mu)] ; \quad (122)$$

and the substitution of Eq. (122) into (18), after some algebra, yields an explicit expression of the gravitational potential as:

$$\mathcal{V}_G = 4\pi G \sum (\lambda_w) \alpha_u^2 \left\{ \sum_{\ell=0}^{+\infty} \left[D_{2\ell} \xi_u^{2\ell} + \frac{\overline{D}_{2\ell}}{\xi_u^{2\ell+1}} \right] P_{2\ell}(\mu) - \frac{1}{6} \Lambda_u \xi_u^2 [1 - P_2(\mu)] + c_{b,u}^\dagger \right\} ; \quad (123)$$

where $c_{b,u}^\dagger$ is defined by Eq. (349) as shown in Appendix B.

The radial component of the gravitational force via Eqs. (12)-(13) reads:

$$\frac{\partial \mathcal{V}_G}{\partial r} = \frac{1}{\alpha_u} \frac{\partial \mathcal{V}_G}{\partial \xi_u} = 4\pi G \sum (\lambda_w) \alpha_u \left\{ \sum_{\ell=0}^{+\infty} \left[2\ell D_{2\ell} \xi_u^{2\ell-1} - \frac{(2\ell+1) \overline{D}_{2\ell}}{\xi_u^{2\ell+2}} \right] P_{2\ell}(\mu) - \frac{1}{3} \Lambda_u \xi_u [1 - P_2(\mu)] \right\} ; \quad (124)$$

on the other hand, the gravitational potential and the radial component of the gravitational force outside the boundary may be approximated by Eqs. (343) and (344), respectively, as shown in Appendix B.

The continuity of the gravitational potential and the radial component of the gravitational force on a selected point of the boundary, $\Xi_u = \Xi_u(\mu)$, makes the terms of same order in Legendre polynomials necessarily be equal in each expression, which implies the following systems of equations:

$$\begin{cases} D_0 + \frac{\overline{D}_0}{\Xi_u} - \frac{1}{6} \Lambda_u \Xi_u^2 + c_{b,u}^\dagger = \frac{c_{0,u}}{\Xi_u} ; \\ -\frac{\overline{D}_0}{\Xi_u^2} - \frac{1}{3} \Lambda_u \Xi_u = -\frac{c_{0,u}}{\Xi_u^2} ; \end{cases} \quad (125)$$

$$\begin{cases} D_2 \Xi_u^2 + \frac{\overline{D}_2}{\Xi_u^3} + \frac{1}{6} \Lambda_u \Xi_u^2 = \frac{c_{2,u}}{\Xi_u^3} ; \\ 2D_2 \Xi_u - \frac{3\overline{D}_2}{\Xi_u^4} + \frac{1}{3} \Lambda_u \Xi_u = -\frac{3c_{2,u}}{\Xi_u^4} ; \end{cases} \quad (126)$$

$$\begin{cases} D_{2\ell} \Xi_u^{2\ell} + \frac{\overline{D}_{2\ell}}{\Xi_u^{2\ell+1}} = \frac{c_{2\ell,u}}{\Xi_u^{2\ell+1}} ; \\ 2\ell D_{2\ell} \Xi_u^{2\ell-1} - \frac{(2\ell+1) \overline{D}_{2\ell}}{\Xi_u^{2\ell+2}} = -\frac{(2\ell+1) c_{2\ell,u}}{\Xi_u^{2\ell+2}} ; \end{cases} \quad (127)$$

and related solutions read:

$$D_0 = \frac{1}{2} \Lambda_u \Xi_u^2 - c_{b,u}^\dagger ; \quad \overline{D}_0 = c_{0,u} - \frac{1}{3} \Lambda_u \Xi_u^3 ; \quad (128)$$

$$D_2 = -\frac{1}{6} \Lambda_u ; \quad \overline{D}_2 = c_{2,u} ; \quad (129)$$

$$D_{2\ell} = 0 ; \quad \overline{D}_{2\ell} = c_{2\ell,u} ; \quad 2\ell > 2 \quad (130)$$

accordingly, Eq. (122) reduces to:

$$\theta_u(\xi_u, \mu) = D_0 + \sum_{\ell=0}^{+\infty} \frac{\overline{D}_{2\ell}}{\xi_u^{2\ell+1}} P_{2\ell}(\mu) + \frac{1}{6} v_u \xi_u^2 [1 - P_2(\mu)] - \frac{1}{6} \Lambda_u \xi_u^2 ; \quad (131)$$

where the constants, $\overline{D}_{2\ell}$, remain to be explicitly determined.

The comparison of Eq. (122) with its counterpart expressed in terms of the EC2 associated functions, Eq. (52), by equating the terms of same degree in Legendre polynomials, yields:

$$A_0 \theta_{0,u}(\xi_u) = D_0 + \frac{\overline{D}_0}{\xi_u} + \frac{1}{6} (v_u - \Lambda_u) \xi_u^2 ; \quad A_0 = 1 ; \quad (132)$$

$$A_2 \theta_{2,u}(\xi_u) = D_2 \xi_u^2 + \frac{\overline{D}_2}{\xi_u^3} - \frac{1}{6} (v_u - \Lambda_u) \xi_u^2 ; \quad (133)$$

$$A_{2\ell} \theta_{2\ell,u}(\xi_u) = D_{2\ell} \xi_u^{2\ell} + \frac{\overline{D}_{2\ell}}{\xi_u^{2\ell+1}} ; \quad 2\ell > 2 ; \quad (134)$$

and additional relations can be inferred from the continuity of the gravitational potential and the radial component of the gravitational force on a selected point of the interface, $\xi_u^* = \xi_u^*(\mu)$. To this respect, it is sufficient solving the systems of Eqs. (132), (133), (134), and related radial first derivatives on both sides:

$$A_0 \theta'_{0,u}(\xi_u) = -\frac{\overline{D}_0}{\xi_u^2} + \frac{1}{3} (v_u - \Lambda_u) \xi_u ; \quad A_0 = 1 ; \quad (135)$$

$$A_2 \theta'_{2,u}(\xi_u) = 2D_2 \xi_u - \frac{3\overline{D}_2}{\xi_u^4} - \frac{1}{3} (v_u - \Lambda_u) \xi_u ; \quad (136)$$

$$A_{2\ell} \theta'_{2\ell,u}(\xi_u) = 2\ell D_{2\ell} \xi_u^{2\ell-1} - \frac{(2\ell+1)\overline{D}_{2\ell}}{\xi_u^{2\ell+2}} ; \quad 2\ell > 2 ; \quad (137)$$

particularized to $\xi_u = \xi_u^*$, where values on the left-hand side relate to the common region and the coefficients, $A_{2\ell}$, attain the same value for both subsystems, as shown in Appendix A. After some algebra, the result is:

$$D_{2\ell} = \frac{1}{4\ell+1} \frac{1}{(\xi_u^*)^{2\ell}} \left\{ A_{2\ell} \left[(2\ell+1) \theta_{2\ell,u}(\xi_u^*) + \xi_u^* \theta'_{2\ell,u}(\xi_u^*) \right] - (2\ell+3) \frac{\delta_{2\ell,0} - \delta_{2\ell,2}}{6} (v_u - \Lambda_u) (\xi_u^*)^2 \right\} ; \quad (138a)$$

$$\overline{D}_{2\ell} = \frac{(\xi_u^*)^{2\ell+1}}{4\ell+1} \left\{ A_{2\ell} \left[2\ell \theta_{2\ell,u}(\xi_u^*) - \xi_u^* \theta'_{2\ell,u}(\xi_u^*) \right] - (2\ell-2) \frac{\delta_{2\ell,0} - \delta_{2\ell,2}}{6} (v_u - \Lambda_u) (\xi_u^*)^2 \right\} ; \quad (138b)$$

where $D_{2\ell} = 0$, $2\ell > 2$, according to Eq. (130), which implies $A_{2\ell} = 0$, $2\ell > 2$, via Eq. (138a) and, in turn, $\overline{D}_{2\ell} = 0$, $2\ell > 2$, via Eq. (138b). Accordingly, Eq. (131) reduces to:

$$\theta_u(\xi_u, \mu) = D_0 + \frac{\overline{D}_0}{\xi_u} + \frac{\overline{D}_2}{\xi_u^3} P_2(\mu) - \frac{1}{6} \Lambda_u \xi_u^2 + \frac{1}{6} v_u \xi_u^2 [1 - P_2(\mu)] ; \quad (139)$$

where D_0 , \overline{D}_0 , \overline{D}_2 , are expressed by Eq. (138) particularized to $2\ell = 0, 2$, respectively.

Finally, the substitution of Eq. (128) into (138a) yields:

$$c_{b,u}^\dagger = -\theta_{0,u}(\xi_u^*) - \xi_u^* \theta'_{0,u}(\xi_u^*) + \frac{1}{2} v_u (\xi_u^*)^2 + \frac{1}{2} \Lambda_u [\Xi_u^2 - (\xi_u^*)^2] ; \quad (140)$$

and the substitution of Eq. (129) into (138a) after some algebra produces:

$$A_2 = -\frac{5}{6} \frac{v_u (\xi_u^*)^2}{3\theta_{2,u}(\xi_u^*) + \xi_u^* \theta'_{2,u}(\xi_u^*)} ; \quad (141)$$

to be compared with their counterparts related to the boundary, Eqs. (348) and (350) written in Appendix B, respectively.

The EC2 associated functions, $\theta_{0,u}$, $\theta_{2,u}$, expressed by Eqs. (132), (133), respectively, can be expanded in Taylor series of starting point, $\xi_{0,u}$. To this aim, the substitution of Eqs. (332) and (336), $m = 1, 3$, written in Appendix A, into (132) and (133), after some algebra yields:

$$A_0 \theta_{0,u}(\xi_u) = D_0 + \frac{\overline{D}_0}{\xi_{0,u}} \sum_{k=0}^{+\infty} (-1)^k \left(\frac{\xi_u - \xi_{0,u}}{\xi_{0,u}} \right)^k + \frac{1}{6} (v_u - \Lambda_u) \times [\xi_{0,u}^2 + 2\xi_{0,u}(\xi_u - \xi_{0,u}) + (\xi_u - \xi_{0,u})^2] ; \quad A_0 = 1 ; \quad (142)$$

$$A_2 \theta_{2,u}(\xi_u) = \frac{\overline{D}_2}{\xi_{0,u}^3} \sum_{k=0}^{+\infty} (-1)^k \frac{(k+1)(k+2)}{2} \left(\frac{\xi_u - \xi_{0,u}}{\xi_{0,u}} \right)^k - \frac{1}{6} v_u \times [\xi_{0,u}^2 + 2\xi_{0,u}(\xi_u - \xi_{0,u}) + (\xi_u - \xi_{0,u})^2] ; \quad (143)$$

provided $|\xi_u - \xi_{0,u}| < \xi_{0,u}$.

The comparison of Eqs. (142) and (143) with related Taylor series expansions, Eq. (65), yields the explicit expression of the coefficients as:

$$A_0 a_{0,1} = -\frac{\overline{D}_0}{\xi_{0,u}^2} + \frac{1}{6} (v_u - \Lambda_u) 2\xi_{0,u} ; \quad A_0 = 1 ; \quad (144a)$$

$$A_0 a_{0,2} = \frac{\overline{D}_0}{\xi_{0,u}^3} + \frac{1}{6} (v_u - \Lambda_u) ; \quad A_0 = 1 ; \quad (144b)$$

$$A_0 a_{0,k} = (-1)^k \frac{\overline{D}_0}{\xi_{0,u}^{k+1}} ; \quad k > 2 ; \quad A_0 = 1 ; \quad (144c)$$

$$A_2 a_{2,1} = -\frac{3\overline{D}_2}{\xi_{0,u}^4} - \frac{1}{6}v_u 2\xi_{0,u} \ ; \quad (145a)$$

$$A_2 a_{2,2} = \frac{6\overline{D}_2}{\xi_{0,u}^5} - \frac{1}{6}v_u \ ; \quad (145b)$$

$$A_2 a_{2,k} = (-1)^k \frac{(k+1)(k+2)}{2} \frac{\overline{D}_2}{\xi_{0,u}^{k+3}} \ ; \quad k > 2 \ ; \quad (145c)$$

according to Eq. (66), where $a_{2\ell,0} = \theta_{2\ell,u}(\xi_{0,u})$ is also included.

Let $\theta_{0,u}(\Xi_{\text{ex},u}) = \theta_u(\Xi_u, \mu) = \theta_{b,u}$ be the (fictitious) spherical isopycnic surface of the expanded sphere, related to the boundary. By use of Eq. (132), an explicit expression reads:

$$D_0 + \frac{\overline{D}_0}{\Xi_u} + \frac{1}{6}(v_u - \Lambda_u)\Xi_u^2 = \theta_{b,u} \ ; \quad (146)$$

which is a third-degree equation where the scaled radius, $\Xi_{\text{ex},u}$, is the lowest positive solution. In the special case, $\theta_{b,u} = 0$, Eq. (146) via (113) reduces to:

$$D_0 + \frac{\overline{D}_0}{\Xi_u} + \frac{1}{6}v_u \Xi_u^2 = 0 \ ; \quad (147)$$

which, in the nonrotating limit, $v_u \rightarrow 0$, has a single solution, $\Xi_{\text{ex},u} = -\overline{D}_0/D_0$, that has necessarily to be positive, hence $\theta_{0,u}(\xi_u) < 0$ as $\xi_u > \Xi_{\text{ex},u}$.

Concerning the special case, $(n_v, n_u) = (n_v, 1)$, the EC2 equation, Eq. (26), takes the form:

$$\frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \theta}{\partial \xi} \right) + \frac{1}{\xi^2} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial \theta}{\partial \mu} \right] - v = -\theta \ ; \quad (148)$$

where $\theta = \theta_u$, $\xi = \Lambda_u^{1/2} \xi_u$, $v = \Lambda_u^{-1} v_u$, via Eqs. (11), (51). A special integral of Eq. (148) is $\theta^{(p)}(\xi, \mu) = v$.

Similarly, the associated EC2 equations, Eqs. (61)-(63), reduce to:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta_{2\ell}}{d\xi} \right) - \frac{(2\ell+1)2\ell}{\xi^2} \theta_{2\ell} - \delta_{2\ell,0} v = -\theta_{2\ell} \ ; \quad (149)$$

where $\theta_{2\ell} = \theta_{2\ell,u}$ via Eq. (51). A special integral of Eq. (149) is $\theta_{2\ell}^{(p)}(\xi) = \delta_{2\ell,0} v$.

In the case under discussion, the set of associated EC2 equations is exactly equivalent to the EC2 equation. Accordingly, if $A_{2\ell} \theta_{2\ell}(\xi) + \delta_{2\ell,0} v$ are solutions of Eqs. (149), the solution of Eq. (148) reads:

$$\theta(\xi, \mu) = \sum_{\ell=0}^{+\infty} A_{2\ell} \theta_{2\ell}(\xi) P_{2\ell}(\mu) + v \ ; \quad A_0 = 1 \ ; \quad (150)$$

where the sum is the general integral of the associated homogeneous equation.

The solutions of Eqs. (149) are e.g., [4], [20]:

$$A_{2\ell}\theta_{2\ell}(\xi) = D_{2\ell}\hat{J}_{+2\ell+1/2}(\xi) + \overline{D}_{2\ell}\hat{J}_{-2\ell-1/2}(\xi) + \delta_{2\ell,0}v ; \quad (151)$$

$$\hat{J}_{\mp k \mp 1/2}(\xi) = (2k+1)!! \sqrt{\frac{\pi}{2\xi}} J_{\mp k \mp 1/2}(\xi) ; \quad (152)$$

$$(2k+1)!! = (2k+1) \cdot (2k-1) \cdot \dots \cdot 5 \cdot 3 \cdot 1 ; \quad k \geq 0 ; \quad (153)$$

where $J_{\mp k \mp 1/2}(\xi)$ are the Bessel functions of half-integer degree, $\mp k \mp 1/2$ e.g., [37], Chap. 24.

The substitution of Eq. (151) into (150) yields:

$$\theta(\xi, \mu) = \sum_{\ell=0}^{+\infty} \left[D_{2\ell}\hat{J}_{+2\ell+1/2}(\xi) + \overline{D}_{2\ell}\hat{J}_{-2\ell-1/2}(\xi) \right] P_{2\ell}(\mu) + v ; \quad (154)$$

which is an exact solution of Eq. (148).

Related expressions of the gravitational potential and radial component of the gravitational force, via Eqs. (12), (13), (18), (349) written in Appendix B, read:

$$\begin{aligned} \mathcal{V}_G(\xi_u, \mu) = 4\pi G \sum (\lambda_w) \alpha_u^2 \left\{ \sum_{\ell=0}^{+\infty} \left[D_{2\ell}\hat{J}_{+2\ell+1/2}(\Lambda_u^{1/2}\xi_u) \right. \right. \\ \left. \left. + \overline{D}_{2\ell}\hat{J}_{-2\ell-1/2}(\Lambda_u^{1/2}\xi_u) \right] + \frac{v_u}{\Lambda_u} - \frac{1}{6}v_u\xi_u^2[1 - P_2(\mu)] + c_{b,u}^\dagger \right\} ; \quad (155) \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathcal{V}_G}{\partial r} = \frac{1}{\alpha_u} \frac{\partial \mathcal{V}_G}{\partial \xi_u} = 4\pi G \sum (\lambda_w) \alpha_u \left\{ \sum_{\ell=0}^{+\infty} \left[D_{2\ell}\hat{J}'_{+2\ell+1/2}(\Lambda_u^{1/2}\xi_u) \right. \right. \\ \left. \left. + \overline{D}_{2\ell}\hat{J}'_{-2\ell-1/2}(\Lambda_u^{1/2}\xi_u) \right] - \frac{1}{3}v_u\xi_u[1 - P_2(\mu)] \right\} ; \quad (156) \end{aligned}$$

where the prime denotes derivation with respect to ξ_u . On the other hand, the gravitational potential and the radial component of the gravitational force outside the boundary may be approximated by Eqs. (343) and (344), respectively, as shown in Appendix B.

The continuity of the gravitational potential and the radial component of the gravitational force on the interface along a selected direction, $\xi_u^* = \xi_u^*(\mu)$, yields the following relations:

$$A_{2\ell}\theta_{2\ell}(\xi_u^*) = D_{2\ell}\hat{J}_{+2\ell+1/2}(\Lambda_u^{1/2}\xi_u^*) + \overline{D}_{2\ell}\hat{J}_{-2\ell-1/2}(\Lambda_u^{1/2}\xi_u^*) + \delta_{2\ell,0}\Lambda_u^{-1}v_u ; \quad (157)$$

$$A_{2\ell}\theta'_{2\ell}(\xi_u^*) = D_{2\ell}\hat{J}'_{+2\ell+1/2}(\Lambda_u^{1/2}\xi_u^*) + \overline{D}_{2\ell}\hat{J}'_{-2\ell-1/2}(\Lambda_u^{1/2}\xi_u^*) ; \quad (158)$$

where $A_0 = 1$. In terms of the constants, $A_{2\ell}$, $D_{2\ell}$, $\overline{D}_{2\ell}$, Eqs. (157) and (158) make a system of two equations in three unknowns, implying an infinity of solutions.

Conversely, the system of equations:

$$\begin{cases} D_{2\ell}^* \hat{J}_{+2\ell+1/2}(\Lambda_u^{1/2} \xi_u^*) + \overline{D}_{2\ell}^* \hat{J}_{-2\ell-1/2}(\Lambda_u^{1/2} \xi_u^*) = \theta_{2\ell}(\xi_u^*) - \delta_{2\ell,0} \Lambda_u^{-1} v_u ; \\ D_{2\ell}^* \hat{J}'_{+2\ell+1/2}(\Lambda_u^{1/2} \xi_u^*) + \overline{D}_{2\ell}^* \hat{J}'_{-2\ell-1/2}(\Lambda_u^{1/2} \xi_u^*) = \theta'_{2\ell}(\xi_u^*) ; \end{cases} \quad (159)$$

$$D_{2\ell}^* = \frac{D_{2\ell}}{A_{2\ell}} ; \quad \overline{D}_{2\ell}^* = \frac{\overline{D}_{2\ell}}{A_{2\ell}} ; \quad A_0 = 1 ; \quad (160)$$

exhibits a unique solution as:

$$D_{2\ell}^* = \frac{D_{2\ell}^*}{\mathcal{D}^{*(2\ell)}} ; \quad \overline{D}_{2\ell}^* = \frac{\overline{D}_{2\ell}^*}{\mathcal{D}^{*(2\ell)}} ; \quad (161)$$

$$\begin{aligned} \mathcal{D}_{2\ell}^* &= \hat{J}'_{-2\ell-1/2}(\Lambda_u^{1/2} \xi_u^*) \left[\theta_{2\ell}(\xi_u^*) - \delta_{2\ell,0} \Lambda_u^{-1} v_u \right] \\ &\quad - \hat{J}_{-2\ell-1/2}(\Lambda_u^{1/2} \xi_u^*) \theta'_{2\ell}(\xi_u^*) ; \end{aligned} \quad (162)$$

$$\begin{aligned} \overline{\mathcal{D}}_{2\ell}^* &= \hat{J}_{+2\ell+1/2}(\Lambda_u^{1/2} \xi_u^*) \theta'_{2\ell}(\xi_u^*) \\ &\quad - \hat{J}'_{+2\ell+1/2}(\Lambda_u^{1/2} \xi_u^*) \left[\theta_{2\ell}(\xi_u^*) - \delta_{2\ell,0} \Lambda_u^{-1} v_u \right] ; \end{aligned} \quad (163)$$

$$\begin{aligned} \mathcal{D}^{*(2\ell)} &= \hat{J}_{+2\ell+1/2}(\Lambda_u^{1/2} \xi_u^*) \hat{J}'_{-2\ell-1/2}(\Lambda_u^{1/2} \xi_u^*) \\ &\quad - \hat{J}'_{+2\ell+1/2}(\Lambda_u^{1/2} \xi_u^*) \hat{J}_{-2\ell-1/2}(\Lambda_u^{1/2} \xi_u^*) ; \end{aligned} \quad (164)$$

where $\mathcal{D}_{2\ell}^*$, $\overline{\mathcal{D}}_{2\ell}^*$, $\mathcal{D}^{*(2\ell)}$, are determinants appearing in the solutions of the systems, expressed by Eq. (159).

In general, $A_{2\ell} = 0$, $2\ell > 2$, according to Eq. (345) as shown in Appendix B, which implies $D_{2\ell} = 0$, $\overline{D}_{2\ell} = 0$, $2\ell > 2$, via Eqs. (157)-(158). Then the cases of interest are $2\ell = 0, 2$, and using explicit expressions of Bessel functions of half-integer degree, Eqs. (152) can be rewritten as:

$$\hat{J}_{+1/2}(\xi) = \frac{\sin \xi}{\xi} ; \quad \hat{J}_{-1/2}(\xi) = \frac{\cos \xi}{\xi} ; \quad (165)$$

$$\hat{J}_{+5/2}(\xi) = 15 \left[\left(\frac{3}{\xi^2} - 1 \right) \frac{\sin \xi}{\xi} - \frac{3 \cos \xi}{\xi} \right] ; \quad (166)$$

$$\hat{J}_{-5/2}(\xi) = 15 \left[\left(\frac{3}{\xi^2} - 1 \right) \frac{\cos \xi}{\xi} + \frac{3 \sin \xi}{\xi} \right] ; \quad (167)$$

and the substitution of Eqs. (165)-(167) into (159)-(164), after a lot of algebra yields:

$$\mathcal{D}_0^* = -\xi^* \{ [\theta_0(\xi_u^*) - v] [\xi^* \sin \xi^* + \cos \xi^*] + \xi^* \theta'_0(\xi_u^*) \cos \xi^* \} ; \quad (168)$$

$$\overline{\mathcal{D}}_0^* = -\xi^* \{ [\theta_0(\xi_u^*) - v] [\xi^* \cos \xi^* - \sin \xi^*] - \xi^* \theta'_0(\xi_u^*) \sin \xi^* \} ; \quad (169)$$

$$\mathcal{D}^{*(0)} = -\xi^* ; \quad (170)$$

$$\mathcal{D}_2^* = -15 \left\{ \left[\frac{9}{(\xi^*)^2} - 4 \right] \theta_2(\xi_u^*) + \left[\frac{3}{(\xi^*)^2} - 1 \right] \xi^* \theta'_2(\xi_u^*) \right\} \frac{\cos \xi^*}{\xi^*}$$

$$-15 \left\{ \left[\frac{9}{\xi^*} - \xi^* \right] \theta_2(\xi_u^*) + 3\theta_2'(\xi_u^*) \right\} \frac{\sin \xi^*}{\xi^*} ; \quad (171)$$

$$\begin{aligned} \overline{D}_2^* = & +15 \left\{ \left[\frac{9}{(\xi^*)^2} - 4 \right] \theta_2(\xi_u^*) + \left[\frac{3}{(\xi^*)^2} - 1 \right] \xi^* \theta_2'(\xi_u^*) \right\} \frac{\sin \xi^*}{\xi^*} \\ & -15 \left\{ \left[\frac{9}{\xi^*} - \xi^* \right] \theta_2(\xi_u^*) + 3\theta_2'(\xi_u^*) \right\} \frac{\cos \xi^*}{\xi^*} ; \end{aligned} \quad (172)$$

$$D^{*(2)} = -\frac{225}{\xi^*} ; \quad (173)$$

where $\xi^* = \Lambda_u^{1/2} \xi_u^*$, $v = \Lambda_u^{-1} v_u$, via Eqs. (11) and (51). The substitution of Eqs. (168)-(173) into (161) yields the explicit expression of the constants, D_0^* , \overline{D}_0^* , D_2^* , \overline{D}_2^* .

In general, Eq. (151) via (160) may be cast under the form:

$$\theta_{2\ell}(\xi) = D_{2\ell}^* \hat{J}_{+2\ell+1/2}(\xi) + \overline{D}_{2\ell}^* \hat{J}_{-2\ell-1/2}(\xi) + \delta_{2\ell,0} v ; \quad (174)$$

which, in particular, can be determined on a point of the boundary along a selected direction, $\Xi = \Xi(\mu)$, $\Xi = \Lambda_u^{1/2} \Xi_u$, together with related first derivative. Then the substitution of $\theta_{2\ell}(\Xi_u)$, $\theta_{2\ell}'(\Xi_u)$, into Eqs. (345)-(349) written in Appendix B, yields an explicit expression of the constants, $A_{2\ell}$, $c_{2\ell,u}$, $c_{b,u}^\dagger$, $c_{b,u}$. In addition, the knowledge of $D_{2\ell}^*$, $\overline{D}_{2\ell}^*$, $A_{2\ell}$, via Eq. (160) implies the knowledge of $D_{2\ell}$, $\overline{D}_{2\ell}$.

The EC2 associated functions, $\theta_{2\ell}(\xi_u)$, can be expanded in Taylor series of starting point, $\xi_{0,u}$, only in the special case of the singular point, $\xi_{0,u} = 0$, where the convergence radius is infinite. On the other hand, series expansions of the kind considered cannot be used in the case under discussion, due to the presence of divergent terms as $\xi_{0,u} \rightarrow 0$ on the right-hand side of Eq. (174). Accordingly, the coefficients of Taylor series expansions, Eq. (65), cannot be expressed in simpler form with respect to the regression formulae, Eqs. (72), (74). For further details, an interested reader is addressed to the parent paper [6].

Let $\theta_{0,u}(\Xi_{\text{ex},u}) = \theta_u(\Xi_u, \mu) = \theta_{b,u}$ be the (fictitious) spherical isopycnic surface of the expanded sphere, related to the boundary. By use of Eqs. (160), (165) and (174), $2\ell = 0$, an explicit expression reads:

$$D_0 \frac{\sin(\Lambda_u^{1/2} \Xi_u)}{\Lambda_u^{1/2} \Xi_u} + \overline{D}_0 \frac{\cos(\Lambda_u^{1/2} \Xi_u)}{\Lambda_u^{1/2} \Xi_u} + \frac{v_u}{\Lambda_u} = \theta_{b,u} ; \quad (175)$$

which is a transcendental equation where the scaled radius, $\Xi_{\text{ex},u}$, is the lowest positive solution. In the nonrotating limit, $v_u \rightarrow 0$, restricting to $\theta_{b,u} = 0$, Eq. (175) has a single solution, $\Xi_{\text{ex},u} = \Lambda_u^{-1/2} \arctan(-\overline{D}_0/D_0)$, which has necessarily to be positive, hence $\theta_{0,u}(\xi_u) < 0$ as $\xi_u > \Xi_{\text{ex},u}$.

Concerning the special case, $(n_v, n_u) = (n_v, 5)$, the EC2 equation, Eq. (26), takes the form:

$$\frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \theta}{\partial \xi} \right) + \frac{1}{\xi^2} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial \theta}{\partial \mu} \right] - v = -\theta^5 ; \quad (176)$$

where $\theta = \theta_u$, $\xi = \Lambda_u^{1/2} \xi_u$, $v = \Lambda_u^{-1} v_u$, via Eqs. (11), (51). A special integral of Eq. (176) is $\theta^{(p)}(\xi, \mu) = v^{1/5}$. Though both the EC2 equation in the non-rotating limit [14], Chap. IV, §4 and the associated EC2 equations [8], [20], can be integrated analytically, related expressions are cumbersome and of little practical utility, keeping in mind the boundary conditions must be related to the interface instead of the centre in the case under discussion. Then the coefficients of Taylor series expansions, Eq. (65), cannot be expressed in simpler form with respect to the regression formulae, Eqs. (72), (74). For further details, an interested reader is addressed to the parent paper [6].

2.5.2 The common region

With regard to the common region, the system of both the EC2 equations, Eq. (22), and the EC2 associated equations of the same order, Eqs. (90)-(92), for each subsystem, must be integrated by use of Eqs. (23)-(25). The boundary conditions are related to the origin as in EC1 polytropes.

Concerning the special case, $(n_v, n_u) = (n_v, 0)$, the EC2 equation, Eq. (26), reduces to:

$$\frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \theta}{\partial \xi} \right) + \frac{1}{\xi^2} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial \theta}{\partial \mu} \right] - v' = -\theta^n ; \quad (177)$$

$$v' = v - \Lambda_{uv} = \frac{v_v - \Lambda_u}{\Lambda_v} ; \quad -\infty < v' \leq v_v ; \quad (178)$$

where $\theta = \theta_v$, $n = n_v$, $\xi = \Lambda_v^{1/2} \xi_v$, $v = v_v / \Lambda_v$, via Eqs. (11), (51), and the parameter, v' , can be conceived as a generalized distortion including both centrifugal and tidal effects.

Accordingly, the EC2 associated equations can be read as in the case of EC1 polytropes [4], [6] where the distortion parameter equals v' , which can attain both positive, null, and negative values. Related solutions exhibit the same formal expression where, in the case under discussion, the variable, ξ , and the generalized distortion parameter, v' , are to be formulated in terms of ξ_v and v_v , Λ_{uv} , via Eqs. (11) and (178), respectively.

If, in addition, $n_v = 0$, Eq. (22) reduces to its counterpart for EC1 polytropes, as:

$$\frac{1}{\xi_w^2} \frac{\partial}{\partial \xi_w} \left(\xi_w^2 \frac{\partial \theta_w}{\partial \xi_w} \right) + \frac{1}{\xi_w^2} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial \theta_w}{\partial \mu} \right] - v_w = -1 ; \quad (179)$$

which, in turn, coincides with its counterpart related to the noncommon region, $w = u$, Eq. (121), provided $\Lambda_u = 1$ therein. The same holds for the solution of Eq. (179) via (122) and for the expression of the gravitational potential and the radial component of the gravitational force via Eqs. (123) and (124), respectively.

Keeping in mind the boundary conditions, Eq. (23), which implies absence of divergent terms at the origin, the solution of Eq. (177) via (122) reads:

$$\theta_w(\xi_w, \mu) = \sum_{\ell=0}^{+\infty} D_{2\ell} \xi_w^{2\ell} P_{2\ell}(\mu) - \frac{1}{6}(1 - v_w) \xi_w^2 [1 - P_2(\mu)] \quad ; \quad (180)$$

where $D_0 = 1$ via Eq. (23) and, in general, $D_{2\ell}$ is denoted as in dealing with the noncommon region, but related values can be different. Within the current Subsection, it shall be intended unlabeled symbols relate to either region, while symbols used for both the common and noncommon region are denoted by the apex, com, ncm, respectively.

The substitution of Eq. (180) into Eq. (18) after some algebra yields an explicit expression of the gravitational potential, as:

$$\mathcal{V}_G = 4\pi G \sum (\lambda_w) \alpha_w^2 \left\{ 1 + \sum_{\ell=1}^{+\infty} D_{2\ell} \xi_w^{2\ell} P_{2\ell}(\mu) - \frac{1}{6} \xi_w^2 [1 - P_2(\mu)] + c_{b,w}^\dagger \right\} \quad ; \quad (181)$$

where $c_{b,u}^\dagger$ is defined by Eq. (349) as shown in Appendix B.

The radial component of the gravitational force via Eqs. (12)-(13) reads:

$$\begin{aligned} \frac{\partial \mathcal{V}_G}{\partial r} &= \frac{1}{\alpha_w} \frac{\partial \mathcal{V}_G}{\partial \xi_w} \\ &= 4\pi G \sum (\lambda_w) \alpha_w \left\{ \sum_{\ell=1}^{+\infty} 2\ell D_{2\ell} \xi_w^{2\ell-1} P_{2\ell}(\mu) - \frac{1}{3} \xi_w [1 - P_2(\mu)] \right\} ; \end{aligned} \quad (182)$$

on the other hand, the gravitational potential and the radial component of the gravitational force within the noncommon region are expressed by Eqs. (123) and (124), respectively.

The continuity of the gravitational potential and the radial component of the gravitational force on a selected point of the interface, $\xi_u^* = \xi_u^*(\mu)$, makes the terms of same order in Legendre polynomials necessarily be equal in each expression, which implies the following systems of equations:

$$\begin{cases} 1 - \frac{1}{6}(\xi_u^*)^2 + c_{b,u}^\dagger = D_0^{(\text{ncm})} + \frac{\overline{D}_0}{\xi_u^*} - \frac{1}{6}\Lambda_u(\xi_u^*)^2 + c_{b,u}^\dagger \quad ; \\ -\frac{1}{3}\xi_u^* = -\frac{\overline{D}_0}{(\xi_u^*)^2} - \frac{1}{3}\Lambda_u \xi_u^* \quad ; \end{cases} \quad (183)$$

$$\begin{cases} D_2^{(\text{com})}(\xi_u^*)^2 + \frac{1}{6}(\xi_u^*)^2 = D_2^{(\text{ncm})}(\xi_u^*)^2 + \frac{\bar{D}_2}{(\xi_u^*)^3} + \frac{1}{6}\Lambda_u(\xi_u^*)^2 & ; \\ 2D_2^{(\text{com})}\xi_u^* + \frac{1}{3}\xi_u^* = 2D_2^{(\text{ncm})}\xi_u^* - \frac{3\bar{D}_2}{(\xi_u^*)^4} + \frac{1}{3}\Lambda_u\xi_u^* & ; \end{cases} \quad (184)$$

$$\begin{cases} D_{2\ell}^{(\text{com})}(\xi_u^*)^{2\ell} = D_{2\ell}^{(\text{ncm})}(\xi_u^*)^{2\ell} + \frac{\bar{D}_{2\ell}}{(\xi_u^*)^{2\ell+1}} & ; \\ 2\ell D_{2\ell}^{(\text{com})}(\xi_u^*)^{2\ell-1} = 2\ell D_{2\ell}^{(\text{ncm})}(\xi_u^*)^{2\ell-1} - \frac{(2\ell+1)\bar{D}_{2\ell}}{(\xi_u^*)^{2\ell+2}} & ; \end{cases} \quad (185)$$

and related solutions via Eqs. (129)-(130) read:

$$D_0^{(\text{ncm})} = 1 - \frac{1 - \Lambda_u}{2}(\xi_u^*)^2 & ; \quad \bar{D}_0 = \frac{1 - \Lambda_u}{3}(\xi_u^*)^3 & ; \quad (186)$$

$$D_2^{(\text{com})} = -\frac{1}{6} & ; \quad \bar{D}_2 = 0 & ; \quad (187)$$

$$D_{2\ell}^{(\text{com})} = 0 & ; \quad \bar{D}_{2\ell} = 0 & ; \quad 2\ell > 2 & ; \quad (188)$$

which, in the case under discussion, yields a simpler formulation of Eqs. (131), (133), (134), as:

$$\theta_u^{(\text{ncm})}(\xi_u, \mu) = D_0^{(\text{ncm})} + \frac{\bar{D}_0}{\xi_u} - \frac{1}{6}\Lambda_u\xi_u^2 + \frac{1}{6}v_u\xi_u^2[1 - P_2(\mu)] & ; \quad (189)$$

$$A_2\theta_{2,u}^{(\text{ncm})}(\xi_u) = -\frac{1}{6}v_u\xi_u^2 & ; \quad (190)$$

$$A_{2\ell}\theta_{2\ell,u}^{(\text{ncm})}(\xi_u) = 0 & ; \quad 2\ell > 2 & ; \quad (191)$$

and Eq. (132) can also be written explicitly by use of Eq. (186).

The substitution of Eqs. (186), (187)-(188), into (128), (129)-(130), respectively, yields:

$$c_{b,u}^\dagger = -1 + \frac{1}{2}(\xi_u^*)^2 + \frac{1}{2}\Lambda_u[\Xi_u^2 - (\xi_u^*)^2] & ; \quad (192)$$

$$c_{0,u} = \frac{1}{3}(\xi_u^*)^3 + \frac{1}{3}\Lambda_u[\Xi_u^3 - (\xi_u^*)^3] & ; \quad (193)$$

$$c_{2\ell,u} = 0 & ; \quad 2\ell > 0 & ; \quad (194)$$

which completes the explicit formulation of the constants within the noncommon region.

According to Eq. (190), the following choice can be made without loss of generality:

$$\theta_{2,u}^{(\text{ncm})}(\xi_u) = \xi_u^2 & ; \quad (195)$$

$$A_2 = -\frac{1}{6}v_u & ; \quad (196)$$

as shown by the substitution of Eq. (195) into (141) and (345) written in Appendix B.

The combination of Eqs. (180) and (187)-(188), keeping in mind $D_0^{(\text{com})} = 1$, yields:

$$\theta_w(\xi_w, \mu) = 1 - \frac{1 - v_w}{6} \xi_w^2 - \frac{1}{6} v_w \xi_w^2 P_2(\mu) ; \quad (197)$$

and the comparison of Eq. (197), via (180), with its counterpart expressed in terms of the EC2 associated functions, Eq. (78), by equating the terms of same degree in Legendre polynomials, yields:

$$A_0 \theta_{0,w}(\xi_w) = 1 - \frac{1 - v_w}{6} \xi_w^2 ; \quad A_0 = 1 ; \quad (198)$$

$$A_2 \theta_{2,w}(\xi_w) = -\frac{1}{6} v_w \xi_w^2 ; \quad (199)$$

$$\theta_{2\ell,w}(\xi_w) = \xi_w^{2\ell} ; \quad 2\ell > 2 ; \quad (200)$$

which coincide with their counterparts related to EC1 polytropes [4], [6]. The substitution of Eqs. (199) into (138a), $2\ell = 2$, yields Eq. (129), as expected.

According to Eq. (199), the following choice can be made without loss of generality:

$$\theta_{2,u}(\xi_u) = \xi_u^2 ; \quad (201)$$

$$A_2 = -\frac{1}{6} v_u ; \quad (202)$$

$$\theta_{2,v}(\xi_v) = -\frac{1}{6} \frac{v_v}{A_2} \xi_v^2 = \frac{v_v}{v_u} \xi_v^2 ; \quad (203)$$

where Eqs. (201) and (203) have the same formal expression only in the special case of subsystems rotating at the same extent, $v_v = v_u$. The substitution of Eq. (201) and (203) into (351) written in Appendix B yields Eq. (202), as expected.

Finally, the substitution of Eq. (197) into (18) discloses the gravitational potential and the gravitational force are purely radial i.e. independent of both μ and v_w within the common region, in the case under discussion.

The comparison of Eqs. (198) and (199) with related MacLaurin series expansions, Eq. (101), via Eqs. (202) and (327) written in Appendix A yields the explicit expression of the coefficients as:

$$A_0 a_{0,1}^{(w,w)} = 0 ; \quad A_0 a_{0,2}^{(w,w)} = -\frac{1 - v_w}{6} ; \quad A_0 a_{0,k+2}^{(w,w)} = 0 ; \quad k > 0 ; \quad (204)$$

$$A_2 a_{2,1}^{(w,w)} = 0 ; \quad A_2 a_{2,2}^{(w,w)} = -\frac{1}{6} v_w ; \quad A_2 a_{2,k+2}^{(w,w)} = 0 ; \quad k > 0 ; \quad (205)$$

while in the general case, $\xi_{0,w} > 0$, the result via Eq. (332) written in Appendix A is:

$$A_0 a_{0,1}^{(w,w)} = -\frac{1 - v_w}{6} 2\xi_{0,w} ; \quad A_0 a_{0,2}^{(w,w)} = -\frac{1 - v_w}{6} ;$$

$$A_0 a_{0,k+2}^{(w,w)} = 0 ; \quad k > 0 ; \quad (206)$$

$$A_2 a_{2,1}^{(w,w)} = -\frac{1}{6} v_w 2\xi_{0,w} ; \quad A_2 a_{2,2}^{(w,w)} = -\frac{1}{6} v_w ; \quad A_2 a_{2,k+2}^{(w,w)} = 0 ; \quad k > 0 ; \quad (207)$$

according to Eq. (94), where $a_{2\ell,0}^{(w,w)} = \theta_{2\ell,w}(\xi_{0,w})$ is also included and, in any case, the convergence radius is infinite. In addition, Eqs. (144) and (145) can be written explicitly via (186) and (187), respectively.

Let $\theta_{0,v}(\Xi_{\text{ex},v}) = \theta_v(\Xi_v, \mu) = \theta_{b,v}$ be the (fictitious) spherical isopycnic surface of the expanded sphere, related to the interface. By use of Eq. (198), an explicit expression reads:

$$1 - \frac{1 - v_v}{6} \xi_v^2 = \theta_{b,v} ; \quad (208)$$

which has a unique (acceptable) solution as:

$$\Xi_{\text{ex},v} = \left[\frac{6(1 - \theta_{b,v})}{1 - v_v} \right]^{1/2} ; \quad (209)$$

in the special case, $\theta_{b,v} = 0$, Eq. (209) reduces to its counterpart related to EC1 polytropes [4], [6].

Concerning the special case, $(n_v, n_u) = (1, 1)$, the EC2 equation, Eq. (22), via (32) takes the form:

$$\begin{aligned} \frac{1}{\xi_u^2} \frac{\partial}{\partial \xi_u} \left(\xi_u^2 \frac{\partial \theta_u}{\partial \xi_u} \right) + \frac{1}{\xi_u^2} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial \theta_u}{\partial \mu} \right] = -(\Lambda_u + \Gamma_{uv} \Lambda_v) \theta_u \\ - \Gamma_{uv} \Lambda_v \frac{v_v - v_u}{6} \xi_u^2 [1 - P_2(\mu)] + v_u - \Lambda_v (1 - \Gamma_{uv}) ; \end{aligned} \quad (210)$$

where the indexes, u, v , can freely be exchanged one with respect to the other.

Let a new parameter be defined as:

$$B_u^2 = \Lambda_u + \Gamma_{uv} \Lambda_v ; \quad (211)$$

where, by use of Eqs. (11) and (33), the following identities can be verified after little algebra:

$$\frac{B_u^2}{B_v^2} = \frac{\alpha_u^2}{\alpha_v^2} = \Gamma_{uv} ; \quad (212)$$

$$\frac{\Lambda_u}{B_u^2} + \frac{\Lambda_v}{B_v^2} = 1 ; \quad (213)$$

$$(B_u^2 - \Lambda_u)(B_v^2 - \Lambda_v) = \Lambda_u \Lambda_v ; \quad (214)$$

accordingly, Eq. (210) can be written under the equivalent form:

$$\frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \theta_u}{\partial \xi} \right) + \frac{1}{\xi^2} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial \theta_u}{\partial \mu} \right] = -\theta_u + 1 - \frac{1 - v_u}{B_u^2} - \frac{\Gamma_{uv} \Lambda_v}{B_u^4} \frac{v_v - v_u}{6} \xi^2 [1 - P_2(\mu)] ; \quad (215)$$

$$\xi = B_u \xi_u = B_v \xi_v ; \quad (216)$$

in the limit of a vanishing v subsystem, $\Lambda_v \rightarrow 0$, $\Lambda_u \rightarrow 1$, $B_u \rightarrow 1$, and Eq. (215) reduces to its counterpart related to EC1 polytropes [4].

Similarly, the associated EC2 equations, Eqs. (90)-(92), via Eqs. (327)-(328) written in Appendix A reduce to:

$$\begin{aligned} \frac{1}{\xi^2} \frac{d}{d\xi} \left[\xi^2 \frac{d(A_{2\ell} \theta_{2\ell,u})}{d\xi} \right] - \frac{(2\ell + 1)2\ell}{\xi^2} A_{2\ell} \theta_{2\ell,u} - \delta_{2\ell,0} v' = -A_{2\ell} \theta_{2\ell,u} \\ + \delta_{2\ell,0} \left[1 - \frac{1}{B_u^2} - \frac{\Gamma_{uv} \Lambda_v}{B_u^4} \frac{v_v - v_u}{6} \xi^2 \right] + \delta_{2\ell,2} \frac{\Gamma_{uv} \Lambda_v}{B_u^4} \frac{v_v - v_u}{6} \xi^2 ; \end{aligned} \quad (217)$$

$$v' = \frac{v_u}{B_u^2} = \frac{v_u}{\Lambda_u \Lambda_u + \Gamma_{uv} \Lambda_v} = \frac{v \Lambda_u}{\Lambda_u + \Gamma_{uv} \Lambda_v} ; \quad (218)$$

where v' has to be conceived as a generalized distortion including both centrifugal and tidal effects.

In the case under discussion, the set of associated EC2 equations is exactly equivalent to the EC2 equation. Accordingly, if $A_{2\ell} \theta_{2\ell,u}(\xi)$ are solutions of Eqs. (217), the solution of Eq. (215) reads:

$$\theta_u(\xi, \mu) = \sum_{\ell=0}^{+\infty} A_{2\ell} \theta_{2\ell,u}(\xi) P_{2\ell}(\mu) ; \quad A_0 = 1 ; \quad (219)$$

on the other hand, the solutions of Eqs. (217) are e.g., [4], [20]:

$$A_{2\ell} \theta_{2\ell,u}(\xi) = D_{2\ell} \hat{J}_{+2\ell+1/2}(\xi) + \overline{D}_{2\ell} \hat{J}_{-2\ell-1/2}(\xi) + (\delta_{2\ell,0} + \delta_{2\ell,2}) \theta_{2\ell,u}^{(p)}(\xi) ; \quad (220)$$

where $\hat{J}_{\mp k \mp 1/2}(\xi)$ are defined by Eq. (152), $\overline{D}_{2\ell} = 0$ via Eq. (88), and $\theta_{2\ell,u}^{(p)}$ are special integrals expressed as:

$$\theta_{0,u}^{(p)}(\xi) = 1 - \frac{1 - v_u}{B_u^2} + \frac{\Gamma_{uv} \Lambda_v}{B_u^4} (v_v - v_u) \left(1 - \frac{1}{6} \xi^2 \right) ; \quad (221)$$

$$\theta_{2,u}^{(p)}(\xi) = \frac{\Gamma_{uv} \Lambda_v}{B_u^4} \frac{v_v - v_u}{6} \xi^2 ; \quad (222)$$

finally, the substitution of Eqs. (220)-(222) into (219) yields:

$$\begin{aligned} \theta_u(\xi_u, \mu) = \sum_{\ell=0}^{+\infty} \left\{ D_{2\ell} \hat{J}_{+2\ell+1/2}(B_u \xi_u) + \delta_{2\ell,0} \left[1 - \frac{1 - v_u}{B_u^2} + \frac{\Gamma_{uv} \Lambda_v}{B_u^4} (v_v - v_u) \right. \right. \\ \left. \left. \times \left(1 - \frac{1}{6} B_u^2 \xi_u^2 \right) \right] + \delta_{2\ell,2} \frac{\Gamma_{uv} \Lambda_v}{B_u^4} \frac{v_v - v_u}{6} B_u^2 \xi_u^2 \right\} P_{2\ell}(\mu) ; \end{aligned} \quad (223)$$

which is an exact solution of Eq. (215). The constant, D_0 , can be inferred from the boundary conditions at the origin via Eqs. (23) and (152), the last implying $\hat{J}_{+2\ell+1/2}(0) = \delta_{2\ell,0}$. After little algebra, the result is:

$$D_0 = \frac{1 - v_u}{B_u^2} - \frac{\Gamma_{uv}\Lambda_v}{B_u^4}(v_v - v_u) \quad ; \quad (224)$$

in the limit of a vanishing v subsystem, $\Lambda_v \rightarrow 0$, $\Lambda_u \rightarrow 1$, $B_u \rightarrow 1$, which yields $D_0 \rightarrow 1 - v_u$, as expected from EC1 polytropes [4].

Related expressions of the gravitational potential and radial component of the gravitational force, via Eqs. (18) and (349) written in Appendix B, read:

$$\begin{aligned} \mathcal{V}_G(\xi_u, \mu) = 4\pi G \sum (\lambda_w) \alpha_u^2 \left\{ \sum_{\ell=0}^{+\infty} \left[D_{2\ell} \hat{J}_{+2\ell+1/2}(B_u \xi_u) \right. \right. \\ \left. \left. + \delta_{2\ell,0} \left[1 - \frac{1 - v_u}{B_u^2} + \frac{\Gamma_{uv}\Lambda_v}{B_u^4}(v_v - v_u) \left(1 - \frac{1}{6} B_u^2 \xi_u^2 \right) \right] \right. \right. \\ \left. \left. + \delta_{2\ell,2} \frac{\Gamma_{uv}\Lambda_v}{B_u^4} \frac{v_v - v_u}{6} B_u^2 \xi_u^2 \right\} P_{2\ell}(\mu) - \frac{1}{6} v_u \xi_u^2 [1 - P_2(\mu)] + c_{b,u}^\dagger \right\} ; \quad (225) \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathcal{V}_G}{\partial r} = \frac{1}{\alpha_u} \frac{\partial \mathcal{V}_G}{\partial \xi_u} = 4\pi G \sum (\lambda_w) \alpha_u \left\{ \sum_{\ell=0}^{+\infty} \left[D_{2\ell} \hat{J}'_{+2\ell+1/2}(B_u \xi_u) \right. \right. \\ \left. \left. - \delta_{2\ell,0} \frac{\Gamma_{uv}\Lambda_v}{B_u^4} (v_v - v_u) \frac{1}{3} B_u^2 \xi_u + \delta_{2\ell,2} \frac{\Gamma_{uv}\Lambda_v}{B_u^4} \frac{v_v - v_u}{3} B_u^2 \xi_u \right\} P_{2\ell}(\mu) \right. \\ \left. - \frac{1}{3} v_u \xi_u [1 - P_2(\mu)] \right\} ; \quad (226) \end{aligned}$$

where the prime denotes derivation with respect to ξ_u . The counterparts of Eqs. (225) and (226), related to the noncommon region, are expressed by Eqs. (155) and (156), respectively.

The continuity of the gravitational potential and the radial component of the gravitational force on the interface along a selected direction, $\xi_u^* = \xi_u^*(\mu)$, via Eqs. (157), (158), (223), (224), implies the following systems of equations:

$$\begin{cases} D_0^{(\text{com})} \left[\hat{J}_{+1/2}(B_u \xi_u^*) - 1 \right] + 1 - \frac{\Gamma_{uv}\Lambda_v}{B_u^2} \frac{v_v - v_u}{6} (\xi_u^*)^2 \\ = D_0^{(\text{ncm})} \hat{J}_{+1/2}(\Lambda_u^{1/2} \xi_u^*) + \overline{D}_0 \hat{J}_{-1/2}(\Lambda_u^{1/2} \xi_u^*) + \frac{v_u}{\Lambda_u} \quad ; \end{cases} \quad (227)$$

$$\begin{cases} D_0^{(\text{com})} \hat{J}'_{+1/2}(B_u \xi_u^*) - \frac{\Gamma_{uv}\Lambda_v}{B_u^2} \frac{v_v - v_u}{3} \xi_u^* \\ = D_0^{(\text{ncm})} \hat{J}'_{+1/2}(\Lambda_u^{1/2} \xi_u^*) + \overline{D}_0 \hat{J}'_{-1/2}(\Lambda_u^{1/2} \xi_u^*) \quad ; \\ D_2^{(\text{com})} \hat{J}_{+5/2}(B_u \xi_u^*) + \frac{\Gamma_{uv}\Lambda_v}{B_u^2} \frac{v_v - v_u}{6} (\xi_u^*)^2 \\ = D_2^{(\text{ncm})} \hat{J}_{+5/2}(\Lambda_u^{1/2} \xi_u^*) + \overline{D}_2 \hat{J}_{-5/2}(\Lambda_u^{1/2} \xi_u^*) \quad ; \\ D_2^{(\text{com})} \hat{J}'_{+5/2}(B_u \xi_u^*) + \frac{\Gamma_{uv}\Lambda_v}{B_u^2} \frac{v_v - v_u}{3} \xi_u^* \\ = D_2^{(\text{ncm})} \hat{J}'_{+5/2}(\Lambda_u^{1/2} \xi_u^*) + \overline{D}_2 \hat{J}'_{-5/2}(\Lambda_u^{1/2} \xi_u^*) \quad ; \end{cases} \quad (228)$$

$$\begin{cases} D_{2\ell}^{(\text{com})} \hat{J}_{+2\ell+1/2}(B_u \xi_u^*) \\ = D_{2\ell}^{(\text{ncm})} \hat{J}_{+2\ell+1/2}(\Lambda_u^{1/2} \xi_u^*) + \overline{D}_{2\ell} \hat{J}_{-2\ell-1/2}(\Lambda_u^{1/2} \xi_u^*) \quad ; \\ D_{2\ell}^{(\text{com})} \hat{J}'_{+2\ell+1/2}(B_u \xi_u^*) \\ = D_{2\ell}^{(\text{ncm})} \hat{J}'_{+2\ell+1/2}(\Lambda_u^{1/2} \xi_u^*) + \overline{D}_{2\ell} \hat{J}'_{-2\ell-1/2}(\Lambda_u^{1/2} \xi_u^*) \quad ; \end{cases} \quad (229)$$

which make explicit expressions of Eqs. (157)-(158).

The substitution of Eq. (224) into (227) yields a system of two equations in two unknowns, $D_0^{(\text{ncm})}$, \overline{D}_0 , the solution of which can be determined using standard methods via Eqs. (159)-(165) and (168)-(170), where $\theta_0(\xi_u^*)$, $\theta'_0(\xi_u^*)$, can be written explicitly. With regard to Eq. (229), $D_{2\ell}^{(\text{ncm})} = 0$, $\overline{D}_{2\ell} = 0$, $2\ell > 2$, via Eqs. (157)-(158) and (345) written in Appendix B, which implies $D_{2\ell}^{(\text{com})} = 0$, $2\ell > 2$. Concerning Eq. (228), $D_2^{(\text{ncm})}$ and \overline{D}_2 can be expressed in terms of $D_2^{(\text{com})}$ by solving related system.

On the other hand, the substitution of Eq. (222) into (220) yields:

$$A_2 \theta_{2,u}^{(\text{com})}(\xi_u) = D_2^{(\text{com})} \left[\hat{J}_{+5/2}(B_u \xi_u) + \frac{\Gamma_{uv} \Lambda_v}{B_u^2 D_2^{(\text{com})}} \frac{v_v - v_u}{6} \xi_u^2 \right] \quad ; \quad (230)$$

where, without loss of generality:

$$\theta_{2,u}^{(\text{com})}(\xi_u) = \hat{J}_{+5/2}(B_u \xi_u) + \frac{\Gamma_{uv} \Lambda_v}{B_u^2 D_2^{(\text{com})}} \frac{v_v - v_u}{6} \xi_u^2 \quad ; \quad (231)$$

$$D_2^{(\text{com})} = A_2 \quad ; \quad (232)$$

and the substitution of Eqs. (231) and (232) into (228) produces:

$$\begin{cases} \hat{J}_{+5/2}(B_u \xi_u^*) + \frac{\Gamma_{uv} \Lambda_v}{B_u^2 A_2} \frac{v_v - v_u}{6} (\xi_u^*)^2 \\ = D_2^* \hat{J}_{+5/2}(\Lambda_u^{1/2} \xi_u^*) + \overline{D}_2^* \hat{J}_{-5/2}(\Lambda_u^{1/2} \xi_u^*) \quad ; \\ \hat{J}'_{+5/2}(B_u \xi_u^*) + \frac{\Gamma_{uv} \Lambda_v}{B_u^2 A_2} \frac{v_v - v_u}{3} \xi_u^* \\ = D_2^* \hat{J}'_{+5/2}(\Lambda_u^{1/2} \xi_u^*) + \overline{D}_2^* \hat{J}'_{-5/2}(\Lambda_u^{1/2} \xi_u^*) \quad ; \end{cases} \quad (233)$$

where D_2^* , \overline{D}_2^* , are defined by Eq. (160) and A_2 , in turn, depends on $D_2^{(\text{ncm})}$, \overline{D}_2 , via Eqs. (157)-(158) and (345) written in Appendix B. Then the system has to be solved through successive iterations in A_2 , selecting an appropriate initial value, up to convergence within an assigned tolerance.

The EC2 associated functions, $\theta_{2\ell,u}(\xi_u)$, via Eqs. (219), (223), (224), after some algebra take the explicit expression:

$$A_0 \theta_{0,u}(\xi_u) = 1 + D_0 \left[\hat{J}_{+1/2}(B_u \xi_u) - 1 \right] - \frac{\Gamma_{uv} \Lambda_v}{B_u^2} \frac{v_v - v_u}{6} \xi_u^2 \quad ; \quad (234)$$

$$A_2 \theta_{2,u}(\xi_u) = D_2 \hat{J}_{+5/2}(B_u \xi_u) + \frac{\Gamma_{uv} \Lambda_v}{B_u^2} \frac{v_v - v_u}{6} \xi_u^2 \quad ; \quad (235)$$

$$A_{2\ell} \theta_{2\ell,u}(\xi_u) = D_{2\ell} \hat{J}_{+2\ell+1/2}(B_u \xi_u) \quad ; \quad 2\ell > 2 \quad ; \quad (236)$$

where $A_{2\ell} = D_{2\ell}$, $\theta_{2\ell,u}(\xi_u) = \hat{J}_{+2\ell+1/2}(B_u \xi_u)$, $2\ell > 2$, with no loss of generality.

The EC2 associated functions, $\theta_{2\ell,u}(\xi_u)$, can be expanded in Taylor series only in the special case of the singular starting point, $\xi_{0,u} = 0$, where the convergence radius is infinite. Restricting to the cases of interest, $2\ell = 0, 2$, the trigonometric functions appearing in Eqs. (234)-(235) via (152) and (165)-(166) can be replaced by corresponding MacLaurin series expansions and, after some algebra, the comparison with related MacLaurin series expansions, Eq. (101), yields:

$$A_0 a_{0,2k+2}^{(u,u)} = (-1)^{k+2} \frac{D_0 B_u^{2k+2}}{(2k+3)!} - \delta_{2k,2} \frac{\Gamma_{uv} \Lambda_v}{B_u^2} \frac{v_v - v_u}{6} ; \quad (237a)$$

$$A_0 a_{0,2k+1}^{(u,u)} = 0 ; \quad 2k \geq 0 ; \quad A_0 = 1 ; \quad (237b)$$

$$A_2 a_{2,2k+2}^{(u,u)} = (-1)^{k+2} 15 \frac{(2k+2)(2k+4)}{(2k+5)!} B_u^{2k+2} + \delta_{2k,2} \frac{\Gamma_{uv} \Lambda_v}{B_u^2} \frac{v_v - v_u}{6} ; \quad (238a)$$

$$A_2 a_{2,2k+1}^{(u,u)} = 0 ; \quad 2k \geq 0 ; \quad (238b)$$

where, unfortunately, the above results cannot be extended to the general case of starting point, $\xi_{0,u} > 0$, and the coefficients of Taylor series expansions, Eq. (93), cannot be expressed in simpler form with respect to the regression formulae, Eqs. (100) and (108). For further details, an interested reader is addressed to the parent paper [6].

Let $\theta_{0,v}(\Xi_{\text{ex},v}) = \theta_v(\Xi_v, \mu) = \theta_{b,v}$ be the (fictitious) spherical isopycnic surface of the expanded sphere, related to the interface. By use of Eqs. (165) and (234), an explicit expression reads:

$$1 + D_0 \left[\frac{\sin(B_v \xi_v)}{B_v \xi_v} - 1 \right] - \frac{\Gamma_{vu} \Lambda_u}{B_v^2} \frac{v_u - v_v}{6} \xi_v^2 = \theta_{b,v} ; \quad (239)$$

which is a transcendental equation where the scaled radius, $\Xi_{\text{ex},v}$, is the lowest positive solution.

Concerning the special case, $(n_v, n_u) = (1, 0)$, the EC2 equation, Eq. (177), reduces to:

$$\frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \theta}{\partial \xi} \right) + \frac{1}{\xi^2} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial \theta}{\partial \mu} \right] - v' = -\theta ; \quad (240)$$

where $\theta = \theta_v$, $\xi = \Lambda_v^{1/2} \xi_v$, $v' = v - \Lambda_{uv}$, $v = v_v / \Lambda_v$, via Eqs. (11), (51), (178), and the parameter, v' , can be conceived as a generalized distortion including both centrifugal and tidal effects.

Keeping in mind the boundary conditions, Eq. (23), which implies absence of divergent terms at the origin, the solution of Eq. (240) via (15) and (154)

reads:

$$\theta(\xi, \mu) = \sum_{\ell=0}^{+\infty} D_{2\ell} \hat{J}_{+2\ell+1/2}(\xi) P_{2\ell}(\mu) + v' \quad ; \quad (241)$$

$$\hat{J}_{+2\ell+1/2}(0) = \delta_{2\ell,0} \quad ; \quad (242)$$

$$D_0 = 1 - v' = \frac{1 - v_v}{\Lambda_v} \quad ; \quad (243)$$

according to the properties of the Bessel functions of half-integer degree e.g., [37], Chap. 24.

In terms of the subsystem, u , Eq. (241) via (32) translates into:

$$\begin{aligned} \theta_u(\xi_u, \mu) = & \Gamma_{vu} \sum_{\ell=0}^{+\infty} D_{2\ell} \hat{J}_{+2\ell+1/2}(\Gamma_{uv}^{1/2} \Lambda_v^{1/2} \xi_u) P_{2\ell}(\mu) + \Gamma_{vu} \frac{v_v - \Lambda_u}{\Lambda_v} + 1 - \Gamma_{vu} \\ & + \frac{v_u - v_v}{6} \xi_u^2 [1 - P_2(\mu)] \quad ; \end{aligned} \quad (244)$$

and related expressions of the gravitational potential and radial component of the gravitational force, via Eqs. (13), (18), (33), and (349) written in Appendix B, read:

$$\begin{aligned} \mathcal{V}_G(\xi_u, \mu) = & 4\pi G \sum (\lambda_w) \alpha_u^2 \left\{ \Gamma_{vu} \sum_{\ell=0}^{+\infty} D_{2\ell} \hat{J}_{+2\ell+1/2}(\Gamma_{uv}^{1/2} \Lambda_v^{1/2} \xi_u) P_{2\ell}(\mu) \right. \\ & \left. + 1 - \Gamma_{vu} + \Gamma_{vu} \frac{v_v - \Lambda_u}{\Lambda_v} - \frac{1}{6} v_v \xi_u^2 [1 - P_2(\mu)] + c_{b,u}^\dagger \right\} ; \end{aligned} \quad (245)$$

$$\begin{aligned} \frac{\partial \mathcal{V}_G}{\partial r} = & \frac{1}{\alpha_u} \frac{\partial \mathcal{V}_G}{\partial \xi_u} = 4\pi G \sum (\lambda_w) \alpha_u \left\{ \Gamma_{vu} \sum_{\ell=0}^{+\infty} D_{2\ell} \hat{J}'_{+2\ell+1/2}(\Gamma_{uv}^{1/2} \Lambda_v^{1/2} \xi_u) P_{2\ell}(\mu) \right. \\ & \left. - \frac{1}{3} v_v \xi_u [1 - P_2(\mu)] \right\} ; \end{aligned} \quad (246)$$

where the prime denotes derivation with respect to ξ_u . The counterparts of Eqs. (245) and (246), related to the noncommon region, are expressed by Eqs. (123) and (124), respectively.

The continuity of the gravitational potential and the radial component of the gravitational force on the interface along a selected direction, $\xi_u^* = \xi_u^*(\mu)$, via Eqs. (123), (124), implies the following systems of equations:

$$\begin{cases} \Gamma_{vu} D_0^{(\text{com})} \hat{J}_{+1/2}(\Gamma_{uv}^{1/2} \Lambda_v^{1/2} \xi_u^*) + \Gamma_{vu} \frac{v_v - \Lambda_u}{\Lambda_v} + 1 - \Gamma_{vu} - \frac{1}{6} v_v (\xi_u^*)^2 + c_{b,u}^\dagger \\ = D_0^{(\text{ncm})} + \frac{\bar{D}_0}{\xi_u^*} - \frac{1}{6} \Lambda_u (\xi_u^*)^2 + c_{b,u}^\dagger \quad ; \\ \Gamma_{vu} D_0^{(\text{com})} \hat{J}'_{+1/2}(\Gamma_{uv}^{1/2} \Lambda_v^{1/2} \xi_u^*) - \frac{1}{3} v_v \xi_u^* = -\frac{\bar{D}_0}{(\xi_u^*)^2} - \frac{1}{3} \Lambda_u \xi_u^* \quad ; \end{cases} \quad (247)$$

$$\begin{cases} \Gamma_{vu} D_2^{(\text{com})} \hat{J}_{+5/2}(\Gamma_{uv}^{1/2} \Lambda_v^{1/2} \xi_u^*) + \frac{1}{6} v_v (\xi_u^*)^2 \\ = D_2^{(\text{ncm})} (\xi_u^*)^2 + \frac{\bar{D}_2}{(\xi_u^*)^3} + \frac{1}{6} \Lambda_u (\xi_u^*)^2 \quad ; \end{cases} \quad (248)$$

$$\begin{cases} \Gamma_{vu} D_2^{(\text{com})} \hat{J}'_{+5/2}(\Gamma_{uv}^{1/2} \Lambda_v^{1/2} \xi_u^*) + \frac{1}{3} v_v \xi_u^* = 2 D_2^{(\text{ncm})} \xi_u^* - \frac{3 \bar{D}_2}{(\xi_u^*)^4} + \frac{1}{3} \Lambda_u \xi_u^* \quad ; \\ \Gamma_{vu} D_{2\ell}^{(\text{com})} \hat{J}_{+2\ell+1/2}(\Gamma_{uv}^{1/2} \Lambda_v^{1/2} \xi_u^*) = D_{2\ell}^{(\text{ncm})} (\xi_u^*)^{2\ell} + \frac{\bar{D}_{2\ell}}{(\xi_u^*)^{2\ell+1}} \quad ; \\ \Gamma_{vu} 2\ell D_{2\ell}^{(\text{com})} \hat{J}'_{+2\ell+1/2}(\Gamma_{uv}^{1/2} \Lambda_v^{1/2} \xi_u^*) = 2\ell D_{2\ell}^{(\text{ncm})} (\xi_u^*)^{2\ell-1} - \frac{(2\ell+1) \bar{D}_{2\ell}}{(\xi_u^*)^{2\ell+2}} \quad ; \end{cases} \quad (249)$$

in three unknowns, $D_{2\ell}^{(\text{com})}$, $D_{2\ell}^{(\text{ncm})}$, $\bar{D}_{2\ell}$.

The substitution of Eq. (243) into (247) yields an explicit expression of $D_0^{(\text{ncm})}$, \bar{D}_0 , as:

$$\bar{D}_0 = (\xi_u^*)^2 \left[-\Gamma_{vu} \frac{1-v_v}{\Lambda_v} \hat{J}'_{+1/2}(\Gamma_{uv}^{1/2} \Lambda_v^{1/2} \xi_u^*) + \frac{v_v - \Lambda_u}{3} \xi_u^* \right] \quad ; \quad (250)$$

$$\begin{aligned} D_0^{(\text{ncm})} &= \Gamma_{vu} \frac{1-v_v}{\Lambda_v} \left[\hat{J}_{+1/2}(\Gamma_{uv}^{1/2} \Lambda_v^{1/2} \xi_u^*) + \xi_u^* \hat{J}'_{+1/2}(\Gamma_{uv}^{1/2} \Lambda_v^{1/2} \xi_u^*) \right] + \Gamma_{vu} \frac{v_v - \Lambda_u}{\Lambda_v} \\ &\quad + 1 - \Gamma_{vu} - \frac{v_v - \Lambda_u}{2} (\xi_u^*)^2 \quad ; \end{aligned} \quad (251)$$

which can be determined via Eq. (165). With regard to Eq. (249), $D_{2\ell}^{(\text{ncm})} = 0$, $\bar{D}_{2\ell} = 0$, $2\ell > 2$, via Eqs. (130) and (139), respectively, which implies $D_{2\ell}^{(\text{com})} = 0$, $2\ell > 2$. Concerning Eq. (248), $D_2^{(\text{ncm})}$ and \bar{D}_2 can be expressed in terms of $D_2^{(\text{com})}$ by solving related system.

On the other hand, the comparison of Eq. (241) with (78), with regard to the terms of second order in Legendre polynomials, yields:

$$A_2 \theta_{2,v}^{(\text{com})}(\xi_v) = D_2^{(\text{com})} \hat{J}_{+5/2}(\Lambda_v^{1/2} \xi_v) \quad ; \quad (252)$$

and the substitution of Eq. (252) into (328) written in Appendix A after some algebra produces:

$$A_2 \theta_{2,u}^{(\text{com})}(\xi_u) = \Gamma_{vu} D_2^{(\text{com})} \hat{J}_{+5/2}(\Gamma_{uv}^{1/2} \Lambda_u^{1/2} \xi_u) + D_2^{(\text{com})} \frac{v_v - v_u}{6 D_2^{(\text{com})}} \xi_u^2 \quad ; \quad (253)$$

where, without loss of generality:

$$\theta_{2,u}^{(\text{com})}(\xi_u) = \Gamma_{vu} \hat{J}_{+5/2}(\Gamma_{uv}^{1/2} \Lambda_u^{1/2} \xi_u) + \frac{v_v - v_u}{6 A_2} \xi_u^2 \quad ; \quad (254)$$

$$D_2^{(\text{com})} = A_2 \quad ; \quad (255)$$

accordingly, Eq. (248) may be cast into the equivalent form:

$$\begin{cases} D_2^*(\xi_u^*)^2 + \frac{\bar{D}_2^*}{(\xi_u^*)^3} = \Gamma_{vu} \hat{J}_{+5/2}(\Gamma_{uv}^{1/2} \Lambda_u^{1/2} \xi_u^*) + \frac{v_v - \Lambda_u}{6 A_2} (\xi_u^*)^2 \quad ; \\ 2 D_2^* \xi_u^* - \frac{3 \bar{D}_2^*}{(\xi_u^*)^4} = \Gamma_{vu} \hat{J}'_{+5/2}(\Gamma_{uv}^{1/2} \Lambda_u^{1/2} \xi_u^*) + \frac{v_v - \Lambda_u}{3 A_2} \xi_u^* \quad ; \end{cases} \quad (256)$$

where D_2^* , \overline{D}_2^* , are defined by Eq. (160) and A_2 , in turn, depends on $D_2^{(\text{ncm})}$, \overline{D}_2 , via Eqs. (253) and (345) written in Appendix B. Then the system has to be solved through successive iterations in A_2 , selecting an appropriate initial value, up to convergence within an assigned tolerance.

Accordingly, Eq. (133) reads:

$$\theta_{2,u}^{(\text{ncm})}(\xi_u) = D_2^* \xi_u^2 + \frac{\overline{D}_2^*}{\xi_u^3} - \frac{v_u - \Lambda_u}{6A_2} \xi_u^2 ; \quad (257)$$

which allows the determination of $\theta_{2,u}(\Xi_u)$, $\theta'_{2,u}(\Xi_u)$, on a selected point of the boundary, $\Xi_u = \Xi_u(\mu)$, and the next iteration value of A_2 via Eq. (345) written in Appendix B.

The EC2 associated functions, $\theta_{2\ell,u}(\xi_u)$, via Eqs. (219), (241), (243), after some algebra take the explicit expression:

$$A_0 \theta_{0,v}(\xi_v) = 1 + \frac{1 - v_v}{\Lambda_v} [\hat{J}_{+1/2}(\Lambda_v^{1/2} \xi_v) - 1] ; \quad A_0 = 1 ; \quad (258)$$

$$A_{2\ell} \theta_{2\ell,v}(\xi_v) = D_{2\ell} \hat{J}_{+2\ell+1/2}(\Lambda_v^{1/2} \xi_v) ; \quad 2\ell > 0 ; \quad (259)$$

where $A_{2\ell} = D_{2\ell}$, $\theta_{2\ell,v}(\xi_v) = \hat{J}_{+2\ell+1/2}(\Lambda_v^{1/2} \xi_v)$, $2\ell > 0$, with no loss of generality. With regard to u subsystem, Eqs. (258)-(259) can be rewritten by use of Eq. (325) appearing in Appendix A. In the limit of a vanishing u subsystem, $\Lambda_u \rightarrow 0$, $\Lambda_v \rightarrow 1$, the associated EC2 functions, $\theta_{2\ell,v}(\xi_v)$, coincide with their counterparts related to EC1 polytropes [4], as expected.

The EC2 associated functions, $\theta_{2\ell,v}(\xi_v)$, can be expanded in Taylor series only in the special case of the singular starting point, $\xi_{0,v} = 0$, where the convergence radius is infinite. Restricting to the cases of interest, $2\ell = 0, 2$, the trigonometric functions appearing in Eqs. (258)-(259) via (152) and (165)-(166) can be replaced by corresponding MacLaurin series expansions and, after some algebra, the comparison with related MacLaurin series expansions, Eq. (101), yields:

$$A_0 a_{0,2k+2}^{(v,v)} = (-1)^{k+2} \frac{1 - v_v}{\Lambda_v} \frac{\Lambda_v^{k+1}}{(2k+3)!} ; \quad (260a)$$

$$A_0 a_{0,2k+1}^{(v,v)} = 0 ; \quad 2k \geq 0 ; \quad A_0 = 1 ; \quad (260b)$$

$$A_2 a_{2,2k+2}^{(v,v)} = (-1)^{k+2} 15 \frac{(2k+2)(2k+4)}{(2k+5)!} \Lambda_v^{k+1} ; \quad (261a)$$

$$A_2 a_{2,2k+1}^{(v,v)} = 0 ; \quad 2k \geq 0 ; \quad (261b)$$

where, unfortunately, the above results cannot be extended to the general case of starting point, $\xi_{0,v} > 0$, and the coefficients of Taylor series expansions,

Eq. (93), cannot be expressed in simpler form with respect to the regression formulae, Eqs. (100) and (108). For further details, an interested reader is addressed to the parent paper [6].

Let $\theta_{0,v}(\Xi_{\text{ex},v}) = \theta_v(\Xi_v, \mu) = \theta_{b,v}$ be the (fictitious) spherical isopycnic surface of the expanded sphere, related to the interface. By use of Eqs. (165) and (258), an explicit expression reads:

$$1 + \frac{1 - v_v}{\Lambda_v} \left[\frac{\sin(\Lambda_v^{1/2} \xi_v)}{\Lambda_v^{1/2} \xi_v} - 1 \right] = \theta_{b,v} ; \quad (262)$$

which is a transcendental equation where the dimensionless radius, $\Xi_{\text{ex},v}$, is the lowest positive solution.

Concerning the special case, $(n_v, n_u) = (0, 1)$, Eq. (177) reduces to (240) where $\theta = \theta_u$, $\xi = \Lambda_u^{1/2} \xi_u$, $v' = v - \Lambda_{vu}$, $v = v_u / \Lambda_u$. Then the solution of Eq. (240) is expressed by Eq. (241) where the constant, D_0 , takes the expression:

$$D_0 = 1 - v' = \frac{1 - v_u}{\Lambda_u} ; \quad (263)$$

according to the properties of the Bessel functions of half-integer degree e.g., [37], Chap. 24.

Related expressions of the gravitational potential and radial component of the gravitational force, via Eqs. (18), (33), and (349) written in Appendix B, read:

$$\begin{aligned} \mathcal{V}_G(\xi_u, \mu) = 4\pi G \sum (\lambda_w) \alpha_u^2 \left\{ \sum_{\ell=0}^{+\infty} D_{2\ell} \hat{J}_{+2\ell+1/2}(\Lambda_u^{1/2} \xi_u) P_{2\ell}(\mu) \right. \\ \left. + \frac{v_u - \Lambda_v}{\Lambda_u} - \frac{1}{6} v_u \xi_u^2 [1 - P_2(\mu)] + c_{b,u}^\dagger \right\} ; \end{aligned} \quad (264)$$

$$\begin{aligned} \frac{\partial \mathcal{V}_G}{\partial r} = \frac{1}{\alpha_u} \frac{\partial \mathcal{V}_G}{\partial \xi_u} = 4\pi G \sum (\lambda_w) \alpha_u \left\{ \sum_{\ell=0}^{+\infty} D_{2\ell} \hat{J}'_{+2\ell+1/2}(\Lambda_u^{1/2} \xi_u) P_{2\ell}(\mu) \right. \\ \left. - \frac{1}{3} v_u \xi_u [1 - P_2(\mu)] \right\} ; \end{aligned} \quad (265)$$

where the prime denotes derivation with respect to ξ_u . The counterparts of Eqs. (264) and (265), related to the noncommon region, are expressed by Eqs. (155) and (156), respectively.

The continuity of the gravitational potential and the gravitational force on the interface along a selected direction, $\xi_u^* = \xi_u^*(\mu)$, via Eqs. (155), (156),

implies the following systems of equations:

$$\begin{cases} D_0^{(\text{com})} \hat{J}_{+1/2}(\Lambda_u^{1/2} \xi_u^*) + \frac{v_u - \Lambda_u}{\Lambda_u} - \frac{1}{6} v_u (\xi_u^*)^2 + c_{b,u}^\dagger \\ = D_0^{(\text{ncm})} \hat{J}_{+1/2}(\Lambda_u^{1/2} \xi_u^*) + \overline{D}_0 \hat{J}_{-1/2}(\Lambda_u^{1/2} \xi_u^*) + \frac{v_u}{\Lambda_u} - \frac{1}{6} v_u (\xi_u^*)^2 + c_{b,u}^\dagger ; \\ D_0^{(\text{com})} \hat{J}'_{+1/2}(\Lambda_u^{1/2} \xi_u^*) - \frac{1}{3} v_u \xi_u^* \\ = D_0^{(\text{ncm})} \hat{J}'_{+1/2}(\Lambda_u^{1/2} \xi_u^*) + \overline{D}_0 \hat{J}'_{-1/2}(\Lambda_u^{1/2} \xi_u^*) - \frac{1}{3} v_u \xi_u^* ; \end{cases} \quad (266)$$

$$\begin{cases} D_{2\ell}^{(\text{com})} \hat{J}_{+2\ell+1/2}(\Lambda_u^{1/2} \xi_u^*) + \delta_{2\ell,2} \frac{1}{6} v_u (\xi_u^*)^2 \\ = D_{2\ell}^{(\text{ncm})} \hat{J}_{+2\ell+1/2}(\Lambda_u^{1/2} \xi_u^*) + \overline{D}_{2\ell} \hat{J}_{-2\ell-1/2}(\Lambda_u^{1/2} \xi_u^*) + \delta_{2\ell,2} \frac{1}{6} v_u (\xi_u^*)^2 ; \\ D_{2\ell}^{(\text{com})} \hat{J}'_{+2\ell+1/2}(\Lambda_u^{1/2} \xi_u^*) + \delta_{2\ell,2} \frac{1}{3} v_u \xi_u^* \\ = D_{2\ell}^{(\text{ncm})} \hat{J}'_{+2\ell+1/2}(\Lambda_u^{1/2} \xi_u^*) + \overline{D}_{2\ell} \hat{J}'_{-2\ell-1/2}(\Lambda_u^{1/2} \xi_u^*) + \delta_{2\ell,2} \frac{1}{3} v_u \xi_u^* ; \end{cases} \quad (267)$$

which, after some algebra, can be merged into a single system as:

$$\begin{cases} \left(D_{2\ell}^{(\text{ncm})} - D_{2\ell}^{(\text{com})} \right) \hat{J}_{+2\ell+1/2}(\Lambda_u^{1/2} \xi_u^*) + \overline{D}_{2\ell} \hat{J}_{-2\ell-1/2}(\Lambda_u^{1/2} \xi_u^*) \\ = -\delta_{2\ell,0} \Lambda_{vu} ; \\ \left(D_{2\ell}^{(\text{ncm})} - D_{2\ell}^{(\text{com})} \right) \hat{J}'_{+2\ell+1/2}(\Lambda_u^{1/2} \xi_u^*) + \overline{D}_{2\ell} \hat{J}'_{-2\ell-1/2}(\Lambda_u^{1/2} \xi_u^*) = 0 ; \end{cases} \quad (268)$$

in the two unknowns, $(D_{2\ell}^{(\text{ncm})} - D_{2\ell}^{(\text{com})})$ and $\overline{D}_{2\ell}$.

In the special case, $2\ell = 0$, the solution of Eq. (268) is determined after some algebra as:

$$\overline{D}_0 = \frac{\Lambda_{vu} \hat{J}'_{+1/2}(\Lambda_u^{1/2} \xi_u^*)}{\mathcal{D}_0^{(0)}} ; \quad (269)$$

$$D_0^{(\text{ncm})} - D_0^{(\text{com})} = -\frac{\Lambda_{vu} \hat{J}'_{-1/2}(\Lambda_u^{1/2} \xi_u^*)}{\mathcal{D}_0^{(0)}} ; \quad (270)$$

$$\mathcal{D}_0^{(0)} = \hat{J}_{+1/2}(\Lambda_u^{1/2} \xi_u^*) \hat{J}'_{-1/2}(\Lambda_u^{1/2} \xi_u^*) - \hat{J}'_{+1/2}(\Lambda_u^{1/2} \xi_u^*) \hat{J}_{-1/2}(\Lambda_u^{1/2} \xi_u^*) ; \quad (271)$$

according to standard methods.

Finally, the substitution of Eq. (263) into (270) yields:

$$D_0^{(\text{ncm})} = -\frac{\Lambda_{vu} \hat{J}'_{-1/2}(\Lambda_u^{1/2} \xi_u^*)}{\mathcal{D}_0^{(0)}} + \frac{1 - v_u}{\Lambda_u} ; \quad (272)$$

and the EC2 associated function, $\theta_{0,u}$, can be explicitly expressed. In the remaining cases, $2\ell > 0$, related systems are made of linearly independent equations. Accordingly, only null solutions exist, hence $D_{2\ell}^{(\text{ncm})} = D_{2\ell}^{(\text{com})}$, $\overline{D}_{2\ell} = 0$. Accordingly, the EC2 associated functions, $\theta_{2\ell,u}$, $2\ell > 0$, maintain their

expression passing from the common to the noncommon region, in the case under discussion.

Related explicit expressions, via Eqs. (78), (241), (263), after some algebra read:

$$A_0 \theta_{0,u}(\xi_u) = 1 + \frac{1 - v_u}{\Lambda_u} \left[\hat{J}_{+1/2}(\Lambda_u^{1/2} \xi_u) - 1 \right] ; \quad A_0 = 1 ; \quad (273)$$

$$A_{2\ell} \theta_{2\ell,u}(\xi_u) = D_{2\ell} \hat{J}_{+2\ell+1/2}(\Lambda_u^{1/2} \xi_u) ; \quad 2\ell > 0 ; \quad (274)$$

where $A_{2\ell} = D_{2\ell}$, $\theta_{2\ell,u}(\xi_u) = \hat{J}_{+2\ell+1/2}(\Lambda_u^{1/2} \xi_u)$, $2\ell > 0$, with no loss of generality, via Eqs. (255) and (259). In the limit of a vanishing v subsystem, $\Lambda_v \rightarrow 0$, $\Lambda_u \rightarrow 1$, the associated EC2 functions, $\theta_{2\ell,u}(\xi_u)$, coincide with their counterparts related to EC1 polytropes [4], as expected.

With regard to Taylor series expansions and dimensionless radius, $\Xi_{\text{ex},u}$, following the same procedure as in the case, $(n_v, n_u) = (1, 0)$, Eqs. (260)-(261) and (262) hold provided the index, v , is replaced by u therein.

Concerning the special case, $(n_v, n_u) = (5, 0)$, the EC2 equation, Eq. (177), reduces to:

$$\frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \theta}{\partial \xi} \right) + \frac{1}{\xi^2} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial \theta}{\partial \mu} \right] - v' = -\theta^5 ; \quad (275)$$

where $\theta = \theta_v$, $\xi = \Lambda_v^{1/2} \xi_v$, $v' = v - \Lambda_{uv}$, $v = v_v / \Lambda_v$, via Eqs. (11), (51), (178), and the parameter, v' , can be conceived as a generalized distortion including both centrifugal and tidal effects.

Similarly, the EC2 associated equations, Eqs. (90)-(92), reduce to:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta_0}{d\xi} \right) - v' = -\theta_0^5 ; \quad (276)$$

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta_2}{d\xi} \right) - \frac{6}{\xi^2} \theta_2 = -5\theta_0^4 \theta_2 \quad (277)$$

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta_{2\ell}}{d\xi} \right) - \frac{2\ell(2\ell+1)}{\xi^2} \theta_{2\ell} = -5\theta_0^4 \theta_{2\ell} ; \quad (278)$$

where $\theta_{2\ell} = \theta_{2\ell,v}$ and the cases of interest, for reasons mentioned above, are $2\ell = 0, 2$.

It can be seen Eqs. (275)-(278) have the same formal expression of their counterparts related to EC1 polytropes [8], keeping in mind $v' \rightarrow 0$ does not necessarily imply spherical isopycnic surfaces. Accordingly, Eqs. (276) and (277) can be integrated analytically provided the generalized distortion, v' , is negligible with respect to the other terms, which implies the inequality:

$$|v'| = \left| \frac{v_v - \Lambda_u}{\Lambda_v} \right| \ll 1 ; \quad (279)$$

where, in particular, $v_v \rightarrow 0$, $\Lambda_u \rightarrow 0$, $\Lambda_v \rightarrow 1$, $\lambda_v \rightarrow +\infty$, that is a Roche model. Related solutions are [8], [20]:

$$\theta_0(\xi) = \cos \nu + \frac{1}{2} v' \tan^2 \nu ; \quad (280)$$

$$\theta_2(\xi) = \frac{15}{128} \times \frac{3[\nu - \sin \nu \cos^3 \nu + \sin^3 \nu \cos \nu] - 8[\sin^3 \nu \cos^5 \nu - \sin^5 \nu \cos^3 \nu]}{\sin^3 \nu \cos^2 \nu} ; \quad (281)$$

$$\nu = \arctan \frac{\xi}{\sqrt{3}} = \arctan \frac{\Lambda_v^{1/2} \xi_v}{\sqrt{3}} = \arctan \frac{\Gamma_{uv}^{1/2} \Lambda_v^{1/2} \xi_u}{\sqrt{3}} ; \quad (282)$$

where the second term on the right-hand side of Eq. (280) can be neglected with respect to the first one via Eq. (279).

With regard to u subsystem, the counterparts of Eqs. (280) and (281) via Eq. (325) written in Appendix A, after some algebra take the expression:

$$\theta_{0,u}(\xi_u) = \Gamma_{vu} \cos \nu + \frac{1}{2} \Gamma_{vu} \frac{v_v - \Lambda_u}{\Lambda_v} \tan^2 \nu + 1 - \Gamma_{vu} + \frac{v_u - v_v}{6} \xi_u^2 ; \quad (283)$$

$$\theta_{2,u}(\xi_u) = \Gamma_{vu} \frac{15}{128} F(\nu) - \frac{v_u - v_v}{6 A_2} \xi_u^2 ; \quad (284)$$

$$F(\nu) = 3F_1(\nu) - 8F_2(\nu) = \frac{\Gamma_{uv} \Lambda_v}{3} \xi_u^2 [3G_1(\nu) - 8G_2(\nu)] ; \quad (285)$$

$$F_1(\nu) = \frac{\nu}{\sin^3 \nu \cos^2 \nu} - \frac{\cos \nu}{\sin^2 \nu} + \frac{1}{\cos \nu} ; \quad (286)$$

$$G_1(\nu) = \frac{\nu}{\sin^5 \nu} - \frac{\cos^3 \nu}{\sin^4 \nu} + \frac{\cos \nu}{\sin^2 \nu} ; \quad (287)$$

$$F_2(\nu) = \cos^3 \nu - \sin^2 \nu \cos \nu ; \quad (288)$$

$$G_2(\nu) = \frac{\cos^5 \nu}{\sin^2 \nu} - \cos^3 \nu ; \quad (289)$$

and related expressions of the gravitational potential and radial component of the gravitational force, via Eqs. (13), (18), (33), and (349) written in Appendix B, read:

$$\mathcal{V}_G(\xi_u, \mu) = 4\pi G \sum (\lambda_w) \alpha_u^2 \left\{ \Gamma_{vu} \left[\cos \nu + \frac{1}{2} \frac{v_v - \Lambda_u}{\Lambda_v} \tan^2 \nu + \frac{15}{128} A_2 F(\nu) P_2(\mu) \right] + 1 - \Gamma_{vu} - \frac{1}{6} v_v \xi_u^2 [1 - P_2(\mu)] + c_{b,u}^\dagger \right\} ; \quad (290)$$

$$\frac{\partial \mathcal{V}_G}{\partial r} = \frac{1}{\alpha_u} \frac{\partial \mathcal{V}_G}{\partial \xi_u} = 4\pi G \sum (\lambda_w) \alpha_u \left\{ \Gamma_{vu} \left[\frac{d \cos \nu}{d \xi_u} + \frac{v_v - \Lambda_u}{\Lambda_v} \tan \nu \frac{d \tan \nu}{d \xi_u} + \frac{15}{128} A_2 \frac{d F}{d \xi_u} P_2(\mu) \right] - \frac{1}{3} v_v \xi_u [1 - P_2(\mu)] \right\} ; \quad (291)$$

where the derivatives via Eq. (282) read:

$$\frac{d \tan \nu}{d \xi_u} = \frac{\tan \nu}{\xi_u} = \frac{\Gamma_{uv} \Lambda_v}{3} \xi_u \frac{\cos \nu}{\sin \nu} ; \quad (292)$$

$$\frac{d \sin \nu}{d \xi_u} = \frac{\sin \nu \cos^2 \nu}{\xi_u} = \frac{\Gamma_{uv} \Lambda_v}{3} \xi_u \frac{\cos^4 \nu}{\sin \nu} ; \quad (293)$$

$$\frac{d \cos \nu}{d \xi_u} = -\frac{\sin^2 \nu \cos \nu}{\xi_u} = -\frac{\Gamma_{uv} \Lambda_v}{3} \xi_u \cos^3 \nu ; \quad (294)$$

$$\frac{d \nu}{d \xi_u} = \frac{\sin \nu \cos \nu}{\xi_u} = \frac{\Gamma_{uv} \Lambda_v}{3} \xi_u \frac{\cos^3 \nu}{\sin \nu} ; \quad (295)$$

$$\begin{aligned} \frac{dF}{d \xi_u} = F'(\xi_u) = \frac{\Gamma_{uv} \Lambda_v}{3} \xi_u \left[\frac{3 \cos \nu}{\sin^4 \nu} - \frac{9 \nu \cos^2 \nu}{\sin^5 \nu} + \frac{6 \nu}{\sin^3 \nu} + \frac{3 \cos^3 \nu}{\sin^2 \nu} + \frac{6 \cos^5 \nu}{\sin^4 \nu} \right. \\ \left. + 3 \cos \nu + 40 \cos^5 \nu - 8 \sin^2 \nu \cos^3 \nu \right] ; \quad (296) \end{aligned}$$

which can be verified after a lot of algebra. The counterparts of Eqs. (290) and (291), related to the noncommon region, are expressed by Eqs. (123) and (124), respectively.

The continuity of the gravitational potential and the gravitational force on the interface along a selected direction, $\xi_u^* = \xi_u^*(\mu)$, via Eqs. (123), (124), implies the following systems of equations:

$$\begin{cases} \Gamma_{vu} \left[\cos \nu^* + \frac{1}{2} \frac{v_v - \Lambda_u}{\Lambda_v} \tan^2 \nu^* \right] + 1 - \Gamma_{vu} - \frac{1}{6} v_v (\xi_u^*)^2 + c_{b,u}^\dagger \\ = D_0 + \frac{\overline{D}_0}{\xi_u^*} - \frac{1}{6} \Lambda_u (\xi_u^*)^2 + c_{b,u}^\dagger ; \end{cases} \quad (297)$$

$$\begin{cases} \Gamma_{vu} \left[\left(\frac{d \cos \nu}{d \xi_u} \right)_{\xi_u^*} + \frac{v_v - \Lambda_u}{\Lambda_v} \tan \nu^* \left(\frac{d \tan \nu}{d \xi_u} \right)_{\xi_u^*} \right] - \frac{1}{3} v_v \xi_u^* = -\frac{\overline{D}_0}{(\xi_u^*)^2} - \frac{1}{3} \Lambda_u \xi_u^* ; \\ \Gamma_{vu} \frac{15}{128} A_2 F(\xi_u^*) + \frac{1}{6} v_v (\xi_u^*)^2 = D_2 (\xi_u^*)^2 + \frac{\overline{D}_2}{(\xi_u^*)^3} + \frac{1}{6} \Lambda_u (\xi_u^*)^2 ; \\ \Gamma_{vu} \frac{15}{128} A_2 F'(\xi_u^*) + \frac{1}{3} v_v \xi_u^* = 2 D_2 \xi_u^* - \frac{3 \overline{D}_2}{(\xi_u^*)^4} + \frac{1}{3} \Lambda_u \xi_u^* ; \end{cases} \quad (298)$$

where $\nu^* = \nu(\xi_u^*)$ and the following relations hold via Eq. (282):

$$\tan^2 \nu = \frac{1}{3} \Gamma_{uv} \Lambda_v \xi_u^2 ; \quad \sin^2 \nu = \frac{\Gamma_{uv} \Lambda_v \xi_u^2}{3 + \Gamma_{uv} \Lambda_v \xi_u^2} ; \quad \cos^2 \nu = \frac{3}{3 + \Gamma_{uv} \Lambda_v \xi_u^2} ; \quad (299)$$

which define the trigonometric functions in terms of ξ_u .

The substitution of Eq. (282) into (297) via (292)-(296), after long but stimulating algebra yields an explicit expression of D_0 , \overline{D}_0 , as:

$$\overline{D}_0 = \frac{1}{3} \Lambda_v (\xi_u^*)^3 \cos^3 \nu^* ; \quad (300)$$

$$D_0 = \Gamma_{vu} \cos^3 \nu^* + 1 - \Gamma_{vu} + \frac{v_u - v_v}{6} (\xi_u^*)^2 ; \quad (301)$$

on the other hand, Eq. (298) may be cast under the equivalent form:

$$\begin{cases} D_2^*(\xi_u^*)^2 + \frac{\overline{D}_2^*}{(\xi_u^*)^3} = \Gamma_{vu} \frac{15}{128} F(\xi_u^*) + \frac{v_v - \Lambda_u}{6A_2} (\xi_u^*)^2 ; \\ 2D_2^* \xi_u^* - \frac{3\overline{D}_2^*}{(\xi_u^*)^4} = \Gamma_{vu} \frac{15}{128} F'(\xi_u^*) + \frac{v_v - \Lambda_u}{3A_2} \xi_u^* ; \end{cases} \quad (302)$$

where D_2^* , \overline{D}_2^* , are defined by Eq. (160) and A_2 , in turn, depends on D_2 , \overline{D}_2 , via Eqs. (133), (136), and (345) written in Appendix B.

Then the system has to be solved through successive iterations in A_2 , selecting an appropriate initial value, up to convergence within an assigned tolerance. With regard to a generic step, after long but stimulating algebra the result is:

$$\begin{aligned} D_2^* = & \frac{1}{5} \frac{15}{128} \frac{\Lambda_v}{3} \left[\frac{15\nu^*}{\sin^3 \nu^*} - \frac{15 \cos \nu^*}{\sin^2 \nu^*} + 30 \cos \nu^* + 40 \cos^3 \nu^* + 48 \cos^5 \nu^* \right] \\ & - \frac{1}{6} \frac{v_u - v_v}{A_2} + \frac{1}{6} \frac{v_v - \Lambda_u}{A_2} ; \end{aligned} \quad (303)$$

$$\begin{aligned} \overline{D}_2^* = & \frac{1}{5} \frac{15}{128} \frac{\Lambda_v}{3} (\xi_u^*)^5 \left[\frac{15\nu^* \cos^2 \nu^*}{\sin^5 \nu^*} - \frac{15 \cos \nu^*}{\sin^4 \nu^*} + \frac{5 \cos \nu^*}{\sin^2 \nu^*} + 10 \cos \nu^* \right. \\ & \left. + 40 \cos^3 \nu^* - 48 \cos^5 \nu^* \right] ; \end{aligned} \quad (304)$$

where \overline{D}_2^* shows no explicit dependence on A_2 . Accordingly, values of $\theta_2(\Xi_u)$, $\theta'_2(\Xi_u)$, on a selected point of the boundary, $\Xi_u = \Xi_u(\mu)$, can be determined via Eq. (257), and the next iteration value of A_2 can be determined via Eq. (345) written in Appendix B.

The EC2 associated functions, $\theta_{0,v}(\xi_v)$, can be expanded in Taylor series only in the special case of the singular starting point, $\xi_{0,v}^\dagger = 0$, restricted to $v' \rightarrow 0$. Accordingly, Eq. (280) via (282) and (299) reduces to:

$$A_0 \theta_{0,v}(\xi_v) = \cos \nu = \left(1 + \frac{1}{3} \Lambda_v \xi_v^2 \right)^{-1/2} ; \quad A_0 = 1 ; \quad (305)$$

where the right-hand side can be expanded in binomial series. The comparison with related MacLaurin series counterpart, Eq. (101), yields:

$$A_0 a_{0,2k+2}^{(v,v)} = (-1)^k \left(\frac{\Lambda_v}{3} \right)^k \frac{1 \cdot 3 \cdot \dots \cdot (2k+1)}{2 \cdot 4 \cdot \dots \cdot (2k+2)} ; \quad (306a)$$

$$A_0 a_{0,2k+1}^{(v,v)} = 0 ; \quad 2k \geq 0 ; \quad A_0 = 1 ; \quad (306b)$$

where the convergence radius is $\Delta_C \xi_v = (3/\Lambda_v)^{1/2}$. Unfortunately, the above results cannot be extended to the general case of starting point, $\xi_{0,v} > 0$, and the coefficients of Taylor series expansions, Eq. (93), cannot be expressed in

simpler form with respect to the regression formula, Eq.(100). For further details, an interested reader is addressed to the parent paper [6].

Let $\theta_{0,v}(\Xi_{\text{ex},v}) = \theta_v(\Xi_v, \mu) = \theta_{b,v}$ be the (fictitious) spherical isopycnic surface of the expanded sphere, related to the interface. By use of Eq.(280), an explicit expression reads:

$$\cos \nu + \frac{1}{2}v' \tan^2 \nu = \theta_{b,v} ; \quad (307)$$

which is a transcendental equation where the scaled radius, $\Xi_{\text{ex},v}$, is the lowest positive solution. In the limit, $v' \rightarrow 0$, Eq.(307) via (282) and (299) reduces to:

$$\left(\frac{3}{3 + \Lambda_v \xi_v^2} \right)^{1/2} = \theta_{b,v} ; \quad (308)$$

which has a unique (acceptable) solution as:

$$\Xi_{\text{ex},v} = \left[\frac{3(1 - \theta_{b,v}^2)}{\Lambda_v \theta_{b,v}^2} \right]^{1/2} ; \quad (309)$$

that in the special case, $\theta_{b,v} = 0$, $\Lambda_v = 1$, reduces to its counterpart related to EC1 polytropes [4], [6], as expected.

3 A guidance example

To the knowledge of the author, subsystems in hydrostatic equilibrium with intersecting boundaries have never been considered in literature. To this respect, an application shall be presented below restricting to the simplest case, $(n_i, n_j) = (0, 0)$, $(\theta_{b,i}, \theta_{b,j}) = (0, 0)$, which via Eq.(35) implies $\Gamma_{ji} \geq 1$, to be intended as a guidance example.

3.1 Input and output parameters

The equipotential surfaces within the common region are expressed by Eq.(197) which, in the special case of the interface, $\theta_{b,w} = 0$, reduces to:

$$1 - \frac{1 - v_w}{6} \Xi_w^2 - \frac{1}{6} v_w \Xi_w^2 P_2(\mu) = 0 ; \quad (310)$$

and the substitution of Eq.(53) into (310) after some algebra yields:

$$\Xi_w = \sqrt{6} \left[1 - \frac{3}{2} v_w (1 - \mu^2) \right]^{-1/2} ; \quad (311)$$

which is the equation of a spheroid. In particular, the polar ($\mu = 1$) and the equatorial ($\mu = 0$) scaled semiaxes are:

$$\Xi_{p,w} = \sqrt{6} \ ; \quad \Xi_{e,w} = \sqrt{6} \left(1 - \frac{3}{2}v_w\right)^{-1/2} \ ; \quad (312)$$

where $\Xi_{p,j}$ and $\Xi_{e,i}$ are different from their counterparts related to the boundary, in that the boundary is not spheroidal via Eq. (189).

The substitution of Eq. (312) into (311) yields:

$$\alpha_w \Xi_w = \alpha_w \Xi_{p,w} \left[1 - \frac{3}{2}v_w(1 - \mu^2)\right]^{-1/2} \ ; \quad (313)$$

in terms of the radial coordinate, $R = \alpha_w \Xi_w$. Owing to intersecting boundaries, the spheroid is restricted to the polar region for i subsystem and to the equatorial region for j subsystem, as shown in Fig. 1.

The intersection of the two spheroids is defined as $\alpha_i \Xi_i(\hat{\mu}) = \alpha_j \Xi_j(\hat{\mu})$ which, via Eqs. (33) and (313), after some algebra yields:

$$\hat{\mu} = \mp \left[1 - \frac{2}{3} \frac{1 - \Gamma_{ij}}{v_i - \Gamma_{ij}v_j}\right]^{1/2} \ ; \quad (314)$$

accordingly, the interface depends on the fractional scaling radius, $\Gamma_{ji}^{1/2}$, and the rotation parameters, v_i, v_j .

The dimensionless polar semiaxis, $\Xi_{p,i}$, and the dimensionless equatorial semiaxis, $\Xi_{e,j}$, via Eq. (312) read:

$$\Xi_{p,i} = \sqrt{6} \ ; \quad (315)$$

$$\Xi_{e,j} = \sqrt{6} \left(1 - \frac{3}{2}v_j\right)^{-1/2} \ ; \quad (316)$$

while the remaining ones, $\Xi_{p,j}$ and $\Xi_{e,i}$, cannot be explicitly expressed in that they extend outside the interface.

The equipotential surfaces within the noncommon region are expressed by Eq. (189) which, in the special case of the boundary, $\theta_{b,w} = 0$, reduces to:

$$D_{0,w} + \frac{\overline{D}_{0,w}}{\Xi_w} - \frac{1}{6}\Lambda_w \Xi_w^2 + \frac{1}{6}v_w \Xi_w^2 [1 - P_2(\mu)] = 0 \ ; \quad (317)$$

where the constants, $D_{0,w}$, $\overline{D}_{0,w}$, via Eq. (186) read:

$$D_{0,w} = 1 - \frac{1 - \Lambda_w}{2}(\xi_w^*)^2 \ ; \quad \overline{D}_{0,w} = \frac{1 - \Lambda_w}{3}(\xi_w^*)^3 \ ; \quad (318)$$

where $\xi_w^* = \xi_w^*(\hat{\mu})$ has to be selected in the case under discussion to ensure continuity. Accordingly, the boundary depends on the fractional scaling radius,

$\Gamma_{ji}^{1/2}$, the fractional density, $\Lambda_{ji} = \lambda_j/\lambda_i$ via $\Lambda_u = 1/(1 + \Lambda_{vu})$, and the rotation parameters, v_i, v_j .

With regard to a generic point on the equatorial plane, $(\xi_w, 0)$, the substitution of Eqs. (197) and (189), respectively, into (9) via (12), (13), after little algebra yields:

$$v_{\text{eq},w}^{(\text{com})} = \frac{2}{3} ; \quad (319)$$

$$v_{\text{eq},w}^{(\text{ncm})} = \frac{2\Lambda_w \xi_w^3 + 6\overline{D}_{0,w}}{3\xi_w^3} ; \quad (320)$$

according if centrifugal support takes place on the common or noncommon region, respectively. Then $v_w > v_{\text{eq},w}$ implies instability.

The substitution of Eq. (53) into (317) after some algebra yields:

$$\frac{1}{6} \left[\Lambda_w - \frac{3}{2} v_w (1 - \mu^2) \right] \Xi_w^3 - D_{0,w} \Xi_w - \overline{D}_{0,w} = 0 ; \quad (321)$$

and the boundary is defined by the (positive) solution of the above third-degree equation, which matches the intersection between subsystem surfaces. More specifically, $\alpha_i \Xi_i(\mu)$, $\mu \leq \hat{\mu}$, and $\alpha_j \Xi_j(\mu)$, $\mu \geq \hat{\mu}$, have to be considered, as shown in Fig. 1.

The subsystem volume can be inferred from the boundary as [7]:

$$S_w = 4\pi \alpha_w^3 I_{S,w} ; \quad (322)$$

$$I_{S,w} = \int_0^1 d\mu \int_0^{\Xi_w(\mu)} \xi_w^2 d\xi_w = \frac{1}{3} \int_0^{\hat{\mu}} \Xi_w^3(\mu) d\mu + \frac{1}{3} \int_{\hat{\mu}}^1 \Xi_w^3(\mu) d\mu ; \quad (323)$$

where $\hat{\mu} > 0$ is expressed by Eq. (314).

The subsystem mass and mass ratio are:

$$M_w = \lambda_w S_w ; \quad m = \frac{M_j}{M_i} = \frac{\lambda_j S_j}{\lambda_i S_i} = \Lambda_{ji} \Gamma_{ji}^{3/2} \frac{I_{S,j}}{I_{S,i}} ; \quad (324)$$

in the case under discussion of homogeneous configurations, $(n_i, n_j) = (0, 0)$.

In conclusion, the input: fractional scaling radius, $\Gamma_{ji}^{1/2} = \alpha_j/\alpha_i$; fractional density, $\Lambda_{ji} = \lambda_j/\lambda_i$; rotation parameters, v_i, v_j ; implies the output: scaled radius, $\Xi_w(\mu)$; homologous axis ratio, $\eta_r = a_{r,j}/a_{r,i}$; subsystem axis ratio, $\epsilon_w = a_{p,w}/a_{e,w}$; interface, $\Xi_i(\mu)$, $1 \geq \mu \geq \hat{\mu}$, $\Xi_j(\mu)$, $0 \leq \mu \leq \hat{\mu}$; boundary, $\Xi_i(\mu)$, $0 \leq \mu \leq \hat{\mu}$, $\Xi_j(\mu)$, $1 \geq \mu \geq \hat{\mu}$; volume ratio, $s = S_j/S_i$; mass ratio, $m = M_j/M_i$.

3.2 Numerical values

Let the substantially flattened, inner subsystem represent a disk, and the slightly flattened, outer subsystem represent a stellar cluster. Values of output parameters for different choices of input parameters are listed in Tables 1

and 2. Interfaces and boundaries of related configurations are shown in Fig. 2, where the top left panel is the source of Fig. 1 and the bottom right panel replots case 7 in reduced scale. In particular, cases 1, 3, 4, are characterized by hydrostatic equilibrium, while the contrary holds for cases 2, 5, 6, 7, where centrifugal support is exceeded within the noncommon region related to i subsystem. From a mathematical standpoint, the reason is the following.

In presence of hydrostatic equilibrium, the third-degree equation which describes the boundary, Eq. (317) via (321), has a single real solution within the range, $\Xi_i(\hat{\mu}) \leq \Xi_i(\mu) \leq \Xi_i(0) = \Xi_{e,i}$, and the surface can be defined as shown in panels 1, 3, 4, of Fig. 2. Conversely, when centrifugal support is exceeded, there are three real solutions among which at least two are positive, getting closer and closer as μ decreases, up to coincide (via a null discriminant) at $\mu = \mu_0 > 0$, and the surface can be defined up to this point (transition from full to dotted lines) as shown in panels 2, 5, 6, 7, of Fig. 2. The analytical continuation of the surface can be traced via the second positive solution up to $\mu = \hat{\mu}$, as shown by dotted lines. Scaled equatorial semiaxes, $\Xi_{e,i}$, subsystem axis ratios, ϵ_i , homologous axis ratios, η_e , volume integrals, $I_{S,i}$, fractional volumes, s , and fractional masses, m , are calculated up to $\mu = \mu_0$ when centrifugal support is exceeded.

Further inspection of Tables 1-2 and Fig. 2 discloses the following main features.

- With respect to the analytical continuation of the interface (dashed curves), the boundary appears less flattened in the polar region and more flattened in the equatorial region.
- Decreasing the fractional scaling radius, $\Gamma_{ji}^{1/2} = \alpha_j/\alpha_i$, yields more flattened configurations, lower fractional volume, $s = S_j/S_i$, lower fractional mass, $m = M_j/M_i$, and vice versa (cases 3 and 1).
- Increasing the fractional central density, $\Lambda_{ji} = \lambda_j/\lambda_i$, yields more flattened configurations, lower s , larger m , and vice versa (cases 4 and 1).
- Increasing the rotation parameter, v_j , yields more flattened j configurations, less flattened i configurations, lower s , lower m , and vice versa (cases 6 and 1).
- Increasing the rotation parameter, v_i , yields more flattened configurations, larger s , larger m , and vice versa (cases 1 and 7).

Caution is needed for configurations where centrifugal support is exceeded (case number with asterisk in Tables 1-2), in that the boundary has been truncated at $\mu = \mu_0$, where the discriminant of related third-degree equation is null (transition from full to dotted lines in Fig. 2).

Table 1: Input and output parameters (prm) for different configurations of EC2 polytropes where $(n_i, n_j) = (0, 0)$ and $(\theta_{b,i}, \theta_{b,j}) = (0, 0)$. Input: fractional scaling radius, $\Gamma_{ji}^{1/2} = \alpha_j/\alpha_i$; fractional density, $\Lambda_{ji} = \lambda_j/\lambda_i$; rotation parameters, v_i, v_j . Output: scaled equatorial semiaxis, $\Xi_{e,w} = a_{e,w}/\alpha_w$; scaled polar semiaxis, $\Xi_{p,w} = a_{p,w}/\alpha_w$ ($\Xi_{p,i} = \sqrt{6} \approx 2.4495$ in all cases); cosine of polar angle at surface intersection, $\hat{\mu} = \cos \hat{\delta}$; subsystem axis ratio, $\epsilon_w = a_{p,w}/a_{e,w}$; homologous axis ratio, $\eta_r = a_{r,j}/a_{r,i}$; volume integral, $I_{S,w}$; fractional volume, $s = S_j/S_i$; fractional mass, $m = M_j/M_i$. For open boundaries (marked by asterisks on the case number), $\Xi_{e,i}$ and related quantities, $\epsilon_i, \eta_e, I_{S,i}, s, m$, are determined on the point which is nearer to the equatorial plane i.e. where μ attains the minimum value. See text for further details.

case:	1	2*	3	4
prm				
$\Gamma_{ji}^{1/2}$	1.6000E+0	1.4000E+0	1.7000E+0	1.6000E+0
Λ_{ji}	1.0000E+0	1.0000E+0	1.0000E+0	5.0000E-1
v_j	2.4000E-1	2.4000E-1	2.4000E-1	2.4000E-1
v_i	5.2000E-1	5.2000E-1	5.2000E-1	5.2000E-1
$\Xi_{e,j}$	3.0619E+0	3.0619E+0	3.0619E+0	3.0619E+0
$\hat{\mu}$	2.1661E-1	4.2266E-1	4.7088E-2	2.1661E-1
$\Xi_{p,j}$	2.5314E+0	2.5065E+0	2.5413E+0	2.5498E+0
$\Xi_{e,i}$	5.4359E+0	5.3735E+0	5.2226E+0	5.3158E+0
ϵ_j	8.2676E-1	8.1863E-1	8.2999E-1	8.3276E-1
ϵ_i	4.5061E-1	4.5585E-1	4.6902E-1	4.6079E-1
η_e	9.0123E-1	7.9773E-1	9.9666E-1	9.2159E-1
η_p	1.6535E+0	1.4326E+0	1.7637E+0	1.6655E+0
$I_{S,j}$	1.9293E+1	1.5479E+1	2.2580E+1	1.9399E+1
$I_{S,i}$	4.2964E+1	1.9719E+1	2.6722E+1	4.2100E+1
s	1.8393E+0	2.1539E+0	4.1515E+0	1.8874E+0
m	1.8393E+0	2.1539E+0	4.1515E+0	9.4370E-1

Table 2: Continuation of Table 1. The reference case, 1, has been repeated to facilitate comparison.

case:	1	5*	6*	7*
prm				
$\Gamma_{ji}^{1/2}$	1.6000E+0	1.6000E+0	1.6000E+0	1.6000E+0
Λ_{ji}	1.0000E+0	2.0000E+0	1.0000E+0	1.0000E+0
v_j	2.4000E-1	2.4000E-1	1.4000E-1	2.4000E-1
v_i	5.2000E-1	5.2000E-1	5.2000E-1	6.6000E-1
$\Xi_{e,j}$	3.0619E+0	3.0619E+0	2.7559E+0	3.0619E+0
$\hat{\mu}$	2.1661E-1	2.1661E-1	3.5627E-1	5.3156E-1
$\Xi_{p,j}$	2.5314E+0	2.5097E+0	2.4685E+0	2.4910E+0
$\Xi_{e,i}$	5.4359E+0	5.5473E+0	5.5083E+0	5.6401E+0
ϵ_j	8.2676E-1	8.1966E-1	8.9573E-1	8.1354E-1
ϵ_i	4.5061E-1	4.4156E-1	4.4469E-1	4.3430E-1
η_e	9.0123E-1	8.8313E-1	8.0050E-1	8.6860E-1
η_p	1.6535E+0	1.6393E+0	1.6124E+0	1.6271E+0
$I_{S,j}$	1.9293E+1	1.9173E+1	1.3784E+1	1.3647E+1
$I_{S,i}$	4.2964E+1	3.0798E+1	2.3106E+1	1.2196E+1
s	1.8393E+0	2.5499E+0	2.4436E+0	4.5835E+0
m	1.8393E+0	5.0998E+0	2.4436E+0	4.5835E+0

4 Discussion

Similarly to distorted EC1 polytropes [13], [15], distorted EC2 polytropes [10], [5], [7] can be described in terms of solutions of associated EC2 equations, Eqs. (61)-(63) and (90)-(92) for the noncommon and the common region, respectively. The above mentioned solutions can be expanded in MacLaurin series via recursion formulae expressed by Eqs. (106) and (109), up to a convenient point within the convergence radius. Taking that point as starting point, Taylor series expansions can be performed via Eqs. (100), (108), and (72), (74), for the common and the noncommon region, respectively, up to a convenient point within the convergence radius. Then the procedure can be repeated up to the first zero, $\Xi_{\text{ex},w}$, of the associated EC2 function, $\theta_{0,w}$, keeping in mind the convergence radius is steadily decreasing down to zero as $\xi_{0,w} \rightarrow \Xi_{\text{ex},w}$. Accordingly, starting points of series expansions involving $\theta_{0,w} < 0$ should imply $\xi_{0,w} > \Xi_{\text{ex},w}$. In the limit of a vanishing subsystem (other than w), the above results reduce to their counterparts related to EC1 polytropes [6].

In general, the knowledge of the convergence radius, $\Delta_C \xi_w$, related to a Taylor series expansion of starting point, $\xi_{0,w}$, is very important owing not only to convergence in itself but, in addition, to uniform convergence wherein $|\xi_w - \xi_{0,w}| < \Delta_C \xi_w$, which implies the series can be differentiated or integrated term-by-term inside the convergence radius, behaving almost like an analytical function e.g., [36], [29].

EC2 polytropes are a useful tool for the description of large-scale celestial bodies, such as galaxies or galaxy clusters, where a visible baryonic (including leptons) subsystem is embedded within a dark nonbaryonic halo. In most applications, each component is treated separately and the model is a simple superposition of the two matter distributions e.g., [12]. To this respect, EC2 polytropes make a further step in that each subsystem readjusts itself in presence of tidal interaction and, in addition, both collisional and collisionless fluids can be represented within the range of polytropic index, $1/2 \leq n \leq 5$ [39]. Finally, a specific model [34], widely used for the description of galaxies and globular clusters, is a different formulation of EC1 polytropes where $n = 5$ e.g., [11].

With regard to the parent paper [10], the current investigation includes additional points, namely (i) boundaries where the density is not vanishing e.g., [24], Chap. IX, §235, which could be useful in some applications e.g., modelling strange quark stars [16], [25] and computing the total mass of a rotating configuration as a function of the central density [35]; (ii) subsystems with intersecting boundaries, where both fill (different volumes of) the noncommon region, which could be useful for the description of special configurations e.g., a substantially flattened subsystem extending outside a slightly flattened one, mimicing a bulge-disk galaxy; (iii) detailed analysis of a few particular cases,

$(n_v, n_u) = (0, 0), (0, 1), (1, 0), (1, 1), (5, 0)$, which could be useful as guidance examples for better understanding a real situation.

To this respect, it is worth emphasizing the current investigation on the special case, $(n_v, n_u) = (0, 0)$, is restricted to the same approximation used for the general case, aiming to improve comparison. An outline of exact solution can be found in an earlier paper [7]. More specifically, the formal expression of the EC2 function remains the same, but the constant, A_2 , is expressed using a different approximation in comparison to the present paper i.e. spheroidal equipotential surfaces within the noncommon region, which implies the non-linear term on the right-hand side of Eq. (189) is negligible with respect to the others, or $1 - \Lambda_u \ll 1$ via Eq. (186) i.e. a nearly vanishing v subsystem.

Homogeneous, concentric and copolar spheroids with intersecting boundaries, regardless of hydrostatic equilibrium, were considered in an earlier investigation [31]. Due attention should be devoted to equilibrium configurations with intersecting boundaries, even if restricted to toy models, to gain more insight on astrophysical systems such as bulge-disk galaxies and star cluster-accretion disk (including the supermassive black hole) within the central parsec. The guidance example shown in Section 3 has been restricted to $(n_i, n_j) = (0, 0)$ for simplicity, which rules out stellar systems in that collisionless equilibrium fluids need $n_w \geq 1/2$, but $(n_i, n_j) = (1, 1)$, well holds to this respect.

With regard to results listed in Tables 1-2 and plotted in Fig. 2, configurations in hydrostatic equilibrium (cases 1, 3, 4) show the inner subsystem slightly exceeds the outer one along the equatorial plane, the volume of the outer subsystem exceeds the inner one by a factor up to about four, and the mass of the outer subsystem is slightly lower than or exceeds the outer one by a factor up to about four. On the other hand, configurations where hydrostatic equilibrium is broken by centrifugal forces (cases 2, 5, 6, 7), disclose similar results (implying a factor up to about five) but with respect to boundaries truncated at $\mu = \mu_0$ (transition point from full to dotted lines in Fig. 2), where the discriminant of the boundary third-degree equation is null. Accordingly, values of equatorial semiaxis, volume and mass of the inner subsystem have to be considered as lower limits.

Though the model under discussion fails in quantitative predictions, still a correct trend could be expected for resembling astrophysical systems, such as early-type disk galaxies and nuclear star clusters embedding a supermassive black hole and the related inner accretion disk.

5 Conclusion

The theory of EC2 polytropes has been reformulated. The method used in earlier investigations [36], [29], [6] for series expansion of the solution of EC1

equation and related associated equations has been extended to the solution of EC2 equation and related associated equations. In addition, special cases where the solution can be expressed analytically have been considered in detail: $(n_i, n_j) = (0, 0), (0, 1), (1, 0), (1, 1), (5, 0)$. Subsystems with nonvanishing density on the boundary and subsystems with intersecting boundaries have also been included in the general theory, with regard to both collisional and collisionless fluids.

A selected class of configurations has been defined in terms of central densities, λ_w , scaling radii, α_w , rotation parameters, v_w , where $w = i, j$. Accordingly, it has been realized subsystems belong to one among the following states: (1) rotating to a different extent and showing different scaling radius; (2) rotating to a different extent but showing equal scaling radius; (3) rotating to the same extent but showing different scaling radius; (4) rotating to the same extent and showing equal scaling radius.

With regard to the noncommon region, it has been realized the solution of the EC2 equation and associated EC2 equations can be expressed similarly to its counterpart related to EC1 polytropes. It has been recognized the contrary holds for the common region, where the solution of EC2 equation and associated EC2 equations involves both subsystems instead of only one, leaving aside a few special cases.

The main results of the current paper may be summarized in the following points.

- (i) Subsystems with nonvanishing density on the boundary are included in the description, which could be useful in e.g., modelling neutron stars and strange quark stars [16], [25] and computing the total mass of a rotating configuration as a function of the central density [35].
- (ii) Subsystems with intersecting boundaries are included in the description, which could be useful for representing special configurations e.g., a substantially flattened subsystem extending outside a slightly flattened one.
- (iii) Special cases where the results can be expressed analytically are analysed in detail with the addition of a guidance example involving homogeneous configurations for simplicity, but with intersecting boundaries.

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Appendix

A Series expansions within the common region

With regard to the common region, the series expansion of the solution of the EC2 equation, Eq. (78), may be inserted into Eq. (32) and the terms of

different degree in Legendre polynomials may be equated separately. After some algebra, the result is:

$$A_{2\ell,v}\theta_{2\ell,v}(\xi_v) = A_{2\ell,u}\theta_{2\ell,u}(\xi_u) + (1 - \Gamma_{uv})[\delta_{2\ell,0} - A_{2\ell,u}\theta_{2\ell,u}(\xi_u)] \\ + (\delta_{2\ell,0} - \delta_{2\ell,2})\Gamma_{uv}\frac{v_v - v_u}{6}\xi_u^2 ; \quad \xi_w \leq \xi_w^* ; \quad (325)$$

which, related to the EC2 associated functions, is the counterpart of Eq. (32), related to the EC2 functions.

The right-hand side of the EC2 associated equations, Eqs. (85)-(87), is independent of the subsystem under consideration, which implies the same holds for the left-hand side, as it can be verified after a long algebra using Eqs. (11) and (325). The result is:

$$\frac{1}{\xi_u^2} \frac{d}{d\xi_u} \left[\xi_u^2 \frac{d(A_{2\ell,u}\theta_{2\ell,u})}{d\xi_u} \right] - \frac{2\ell(2\ell+1)}{\xi_u^2} A_{2\ell,u}\theta_{2\ell,u} - \delta_{2\ell,0}v_u \\ = \frac{1}{\xi_v^2} \frac{d}{d\xi_v} \left[\xi_v^2 \frac{d(A_{2\ell,v}\theta_{2\ell,v})}{d\xi_v} \right] - \frac{2\ell(2\ell+1)}{\xi_v^2} A_{2\ell,v}\theta_{2\ell,v} - \delta_{2\ell,0}v_v ; \quad (326)$$

where $A_{0,w} = 1$ owing to the boundary conditions, Eqs. (23) and (88). Without loss of generality, the EC2 associated functions, $\theta_{2\ell,w}$, can be normalized to yield coinciding coefficients, $A_{2\ell,w}$, $2\ell > 0$. Accordingly, the following relations hold:

$$A_{0,u} = A_{0,v} = A_0 = 1 ; \quad A_{2\ell,u}(v_u, v_v) = A_{2\ell,v}(v_u, v_v) = A_{2\ell}(v_u, v_v) ; \quad (327)$$

regardless of the subsystem under consideration. Finally, the substitution of Eq. (327) into (325) yields:

$$\theta_{2\ell,v}(\xi_v) = \theta_{2\ell,u}(\xi_u) + (1 - \Gamma_{uv}) \left[\frac{\delta_{2\ell,0}}{A_{2\ell}} - \theta_{2\ell,u}(\xi_u) \right] \\ + \Gamma_{uv} \frac{\delta_{2\ell,0} - \delta_{2\ell,2}}{A_{2\ell}} \frac{v_v - v_u}{6} \xi_u^2 ; \quad \xi_w \leq \xi_w^* ; \quad (328)$$

where the EC2 associated functions, $\theta_{2\ell,v}$, may be expanded in Taylor series as:

$$\theta_{2\ell,v}(\xi_v) = \sum_{k=0}^{+\infty} a_{2\ell,k}^{(v,v)}(\xi_{0,v})(\xi_v - \xi_{0,v})^k = \sum_{k=0}^{+\infty} a_{2\ell,k}^{(v,u)}(\xi_{0,u})(\xi_u - \xi_{0,u})^k ; \quad (329)$$

$$a_{2\ell,0}^{(v,u)}(\xi_{0,u}) = \theta_{2\ell,v}(\xi_{0,u}); \quad a_{2\ell,k}^{(v,u)}(\xi_{0,u}) = \frac{1}{k!} \left(\frac{d^k \theta_{2\ell,v}}{d\xi_u^k} \right)_{\xi_{0,u}} ; \quad (330)$$

and, in particular, $a_{2\ell,1}^{(v,u)}(\xi_{0,u}) = \theta'_{2\ell,v}(\xi_{0,u})$ and $a_{2\ell,2}^{(v,u)}(\xi_{0,u}) = \theta''_{2\ell,v}(\xi_{0,u})/2$. The variables, ξ_u and ξ_v , are linked via Eq. (13). The coefficients belonging to

different series expansions appearing in Eq. (329), via Eqs. (13), (33) and (93), are related as:

$$a_{2\ell,k}^{(v,v)} = \Gamma_{vu}^{k/2} a_{2\ell,k}^{(v,u)} ; \quad (331)$$

with regard to the common region.

The substitution of Eqs. (93) and (329) into (328) by use of the identity:

$$\xi_w^2 = \xi_{0,w}^2 + 2\xi_{0,w}(\xi_w - \xi_{0,w}) + (\xi_w - \xi_{0,w})^2 ; \quad (332)$$

after equating the terms of same degree in $(\xi_w - \xi_{0,w})^k$ yields:

$$a_{2\ell,k}^{(v,u)}(\xi_{0,u}) = \Gamma_{uv} \left\{ a_{2\ell,k}^{(u,u)}(\xi_{0,u}) - \frac{\delta_{0k}\delta_{2\ell,0}}{A_{2\ell}}(1 - \Gamma_{vu}) - \frac{\delta_{2\ell,0} - \delta_{2\ell,2}}{A_{2\ell}} \frac{v_u - v_v}{6} [\delta_{0k}\xi_{0,u}^2 + 2\delta_{1k}\xi_{0,u} + \delta_{2k}] \right\} ; \quad (333)$$

provided both series are convergent.

In the special case, $k = 0$, Eq. (333) via (94) and (330) reduces to Eq. (328) particularized to the starting point, $\xi_w = \xi_{0,w}$. In the special case, $k = 1$, Eq. (333) via (94) and (330) reduces to Eq. (328) particularized to the starting point, $\xi_w = \xi_{0,w}$, after derivation on both sides with respect to ξ_w . Similar considerations hold for generic k .

In the special case of the singular starting point, $\xi_{0,w} = 0$, Eq. (333) reduces to:

$$a_{2\ell,k}^{(v,u)}(0) = \Gamma_{uv} \left[a_{2\ell,k}^{(u,u)}(0) - \frac{\delta_{0k}\delta_{2\ell,0}}{A_{2\ell}}(1 - \Gamma_{vu}) - \delta_{2k} \frac{\delta_{2\ell,0} - \delta_{2\ell,2}}{A_{2\ell}} \frac{v_u - v_v}{6} \right] ; \quad (334)$$

provided both series are convergent.

Finally, Taylor series expansions expressed by Eqs. (71) and (73) are special cases of the power, $1/\xi_w^m$. Aiming to write a general formula, the k -th derivative is:

$$\frac{d^k}{d\xi_w^k} \frac{1}{\xi_w^m} = \frac{(-1)^k m(m+1)\dots(m+k-1)}{\xi_w^{m+k}} ; \quad (335)$$

and related Taylor series expansion after some algebra reads:

$$\frac{1}{\xi_w^m} = \frac{1}{\xi_{0,w}^m} \sum_{k=0}^{+\infty} (-1)^k \binom{k+m-1}{m-1} \left(\frac{\xi_w - \xi_{0,w}}{\xi_{0,w}} \right)^k ; \quad |\xi_w - \xi_{0,w}| < \xi_{0,w} ; \quad (336)$$

where $\xi_{0,w}$ is the starting point. In the special cases, $m = 1, 2$, Eq. (336) reduces to (71), (73), respectively.

B Determination of the coefficients, $A_{2\ell}$

With regard to the outer (along the direction considered) subsystem, u , the substitution of Eqs. (52) and (78) into (18) changes the expression of the gravitational potential and the radial component of the gravitational force as:

$$\mathcal{V}_G = 4\pi G \sum (\lambda_w) \alpha_u^2 \sum_{\ell=0}^{+\infty} \left[A_{2\ell}^{(\text{xxx})} \theta_{2\ell,u}^{(\text{xxx})}(\xi_u) - \frac{\delta_{2\ell,0} - \delta_{2\ell,2}}{6} v_u \xi_u^2 \right] P_{2\ell}(\mu) + \mathcal{V}_{b,u}^\dagger ; \quad (337)$$

$$\frac{d\mathcal{V}_G}{dr} = \frac{1}{\alpha_u} \frac{d\mathcal{V}_G}{d\xi_u} = 4\pi G \sum (\lambda_w) \alpha_u \times \sum_{\ell=0}^{+\infty} \left[A_{2\ell}^{(\text{xxx})} \theta'_{2\ell,u}(\xi_u) - \frac{\delta_{2\ell,0} - \delta_{2\ell,2}}{3} v_u \xi_u \right] P_{2\ell}(\mu) ; \quad (338)$$

$$\mathcal{V}_{b,u}^\dagger = \mathcal{V}_{b,u} - 4\pi G \sum (\lambda_w) \alpha_u^2 \theta_{b,u} ; \quad (339)$$

where $A_0^{(xxx)} = 1$ according to Eqs. (55) and (79), Eq. (52) has been expressed in terms of ξ_u instead of $\xi_{u1} = \Lambda_u^{1/2} \xi_u$, and xxx = com, ncm, according if (ξ_u, μ) belongs to the common region, $0 \leq \xi_u \leq \xi_u^*$, or the noncommon region, $\xi_u^* \leq \xi_u \leq \Xi_u$.

The continuity of the gravitational potential and the radial component of the gravitational force on the interface, $\xi_u = \xi_u^*$, implies the validity of the following relations:

$$A_{2\ell}^{(\text{com})} \theta_{2\ell,u}^{(\text{com})}(\xi_u^*) = A_{2\ell}^{(\text{ncm})} \theta_{2\ell,u}^{(\text{ncm})}(\xi_u^*) ; \quad (340)$$

$$A_{2\ell}^{(\text{com})} \theta'_{2\ell,u}(\xi_u^*) = A_{2\ell}^{(\text{ncm})} \theta'_{2\ell,u}(\xi_u^*) ; \quad (341)$$

where, in general, $\theta'_{2\ell,w} = d\theta_{2\ell,w}/d\xi_w$. On the other hand, the continuity of the first latitudinal component of the gravitational force, $\partial\mathcal{V}_G/\partial\mu$, on the interface, reduces to Eq. (340). In fractional form, Eqs. (340) and (341) translate into:

$$\frac{A_{2\ell}^{(\text{ncm})}}{A_{2\ell}^{(\text{com})}} = \frac{\theta_{2\ell,u}^{(\text{com})}(\xi_u^*)}{\theta_{2\ell,u}^{(\text{ncm})}(\xi_u^*)} = \frac{\theta'_{2\ell,u}(\xi_u^*)}{\theta'_{2\ell,u}(\xi_u^*)} ; \quad (342)$$

or, in other words, $\theta_{2\ell,u}^{(\text{com})}$ and $\theta_{2\ell,u}^{(\text{ncm})}$ on the interface differ by a constant factor at most.

Outside the boundary of a body of revolution, the gravitational potential and the radial component of the gravitational force, via Eqs. (12)-(13), at sufficiently large distances can be expressed as e.g., [27], Chap. VII, §193:

$$\mathcal{V}_G(\xi_u, \mu) = 4\pi G \sum (\lambda_w) \alpha_u^2 \sum_{\ell=0}^{+\infty} \frac{c_{2\ell,u}}{\xi_u^{2\ell+1}} P_{2\ell}(\mu) ; \quad \xi_u \gg \Xi_u ; \quad (343)$$

$$\frac{\partial\mathcal{V}_G}{\partial r} = \frac{1}{\alpha_u} \frac{\partial\mathcal{V}_G}{\partial \xi_u} = 4\pi G \sum (\lambda_w) \alpha_u \sum_{\ell=0}^{+\infty} \frac{-(2\ell+1)c_{2\ell,u}}{\xi_u^{2\ell+2}} P_{2\ell}(\mu) ; \quad (344)$$

where $c_{2\ell,u}$ are dimensionless coefficients and odds terms are ruled out by symmetry with respect to the equatorial plane. Strictly speaking, Eqs. (343)-(344) are exact for spherical-symmetric matter distributions and Roche models [24], Chap. IX §§229-232, but hold to an acceptable extent provided oblateness remains sufficiently small and/or concentration maintains sufficiently high. For further details, an interested reader is addressed to an earlier investigation [9].

The continuity of the gravitational potential and the radial component of the gravitational force on a selected point of the boundary, $\Xi_u = \Xi_u(\mu)$, implies Eqs. (337)-(338) and (343)-(344), respectively, match at (Ξ_u, μ) for the terms of same degree in Legendre polynomials. The result is e.g., [9]:

$$A_{2\ell}^{(\text{ncm})} = -\delta_{2\ell,2} \frac{5}{6} \frac{v_u \Xi_u^2}{3\theta_{2\ell,u}^{(\text{ncm})}(\Xi_u) + \Xi_u \theta'_{2\ell,u}(\Xi_u)} ; \quad 2\ell > 0 ; \quad (345)$$

$$c_{2\ell,u} = \frac{1}{5} A_{2\ell}^{(\text{ncm})} \Xi_u^3 \left[2\theta_{2\ell,u}^{(\text{ncm})}(\Xi_u) - \Xi_u \theta'_{2\ell,u}(\Xi_u) \right] ; \quad 2\ell > 0 ; \quad (346)$$

$$A_0^{(\text{ncm})} = 1 ; \quad c_{0,u} = -\Xi_u^2 \theta'_{0,u}(\Xi_u) + \frac{1}{3} v_u \Xi_u^3 ; \quad (347)$$

$$c_{b,u}^\dagger = \frac{\mathcal{V}_{b,u}^\dagger}{4\pi G \sum (\lambda_w) \alpha_u^2} = -\theta_{0,u}^{(\text{ncm})}(\Xi_u) - \Xi_u \theta'_{0,u}(\Xi_u) + \frac{1}{2} v_u \Xi_u^2 ; \quad (348)$$

$$c_{b,u} = \frac{\mathcal{V}_{b,u}}{4\pi G \sum (\lambda_w) \alpha_u^2} = c_{b,u}^\dagger + \theta_{b,u} ; \quad (349)$$

where a value of Ξ_u on the boundary has necessarily to be fixed for defining the approximation used and calculating the coefficients, $A_{2\ell}^{(\text{ncm})}$, among others. Viable alternatives could be $(\Xi_u, \mu) = (\Xi_{E,u}, 1/\sqrt{3})$ [13] and $(\Xi_u, \mu) = (\Xi_{p,u}, 1)$ [9], where $\Xi_{E,u}$ relates to the nonrotating sphere and $\Xi_{p,u}$ to the rotating configuration, respectively.

With regard to the common region, the sole restriction is from Eq. (342). Accordingly, $A_{2\ell}^{(\text{ncm})} = A_{2\ell}^{(\text{com})} = A_{2\ell}$ without loss of generality, and using Eqs. (327), (345), yields:

$$A_{2\ell} = \delta_{2\ell,0} - \delta_{2\ell,2} \frac{5}{6} \frac{v_u \Xi_u^2}{3\theta_{2\ell,u}^{(\text{ncm})}(\Xi_u) + \Xi_u \theta'_{2\ell,u}(\Xi_u)} ; \quad (350)$$

for a selected point of the boundary, (Ξ_u, μ) .

For specified configurations where $\theta_{2\ell,w}^{(\text{com})}(\xi_w)$ is known via numerical integrations, the particularization of Eq. (328) to $2\ell = 2$ after some algebra yields:

$$A_2 = -\frac{1}{6} \frac{\Gamma_{uv}(v_v - v_u) \xi_u^2}{\theta_{2,v}^{(\text{com})}(\xi_v) - \Gamma_{uv} \theta_{2,u}^{(\text{com})}(\xi_u)} ; \quad (351)$$

where no approximation is involved.

C Corrigendum

With regard to a quoted reference [9], Eq. (76) therein is affected by a printing error, which is propagated to the next Eq. (77). The correct formulation reads:

$$\theta'_2(\xi) = \psi'_2(\xi) = 15 \left[\left(-\frac{9}{\xi^3} + \frac{4}{\xi} \right) \frac{\sin \xi}{\xi} + \left(\frac{9}{\xi^2} - 1 \right) \frac{\cos \xi}{\xi} \right] ; \quad (76)$$

$$A_2 = -\frac{v\Xi^2}{18} \left[\frac{\sin \Xi}{\Xi} - \cos \Xi \right]^{-1} ; \quad (77)$$

respectively. On the other hand, the above mentioned equations were not used in computations, which implies all results reported therein remain unchanged.

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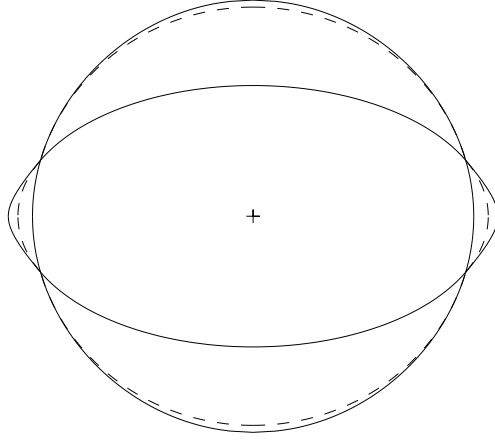


Figure 1: Subsystems with intersecting boundaries, $(n_i, n_j) = (0, 0)$ and $(\theta_{b,i}, \theta_{b,j}) = (0, 0)$. By definition, the inner subsystem, i , has the pole closer to the centre, and the outer subsystem, j , has the pole farther, hence $\alpha_i \Xi_{p,i} \leq \alpha_j \Xi_{p,j}$. The interface lies on two spheroids, where the locus of intersection points is related to the polar angle, $\hat{\delta} = \arccos \hat{\mu}$ (not shown to save clarity). The boundary of the system cannot be fitted by the analytical continuation of the interface (dashed), owing to the occurrence of nonlinear terms in related equations. See text for further details.

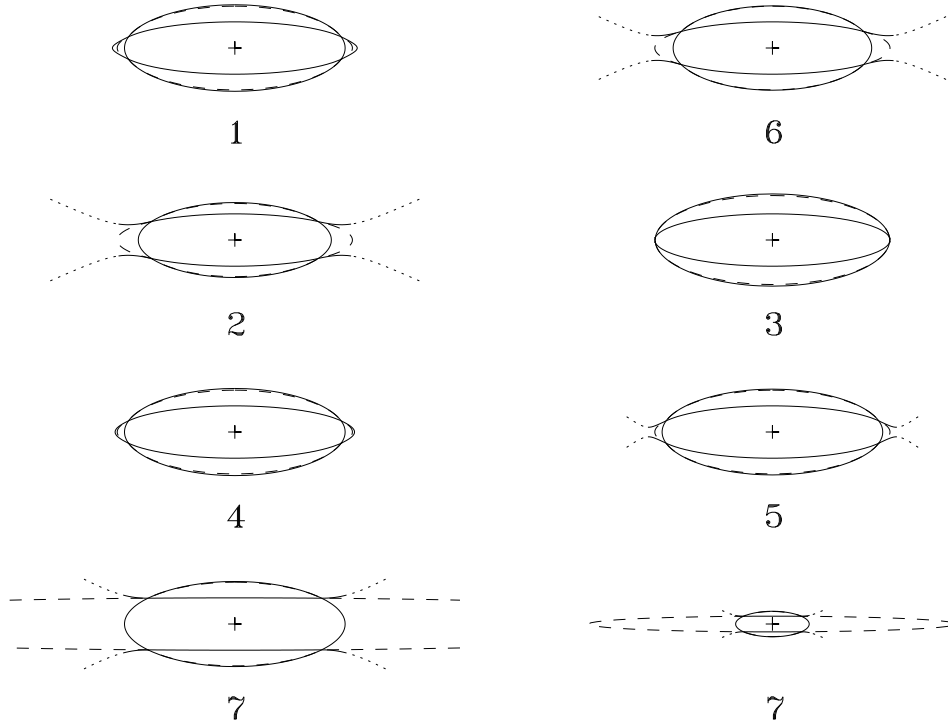


Figure 2: Boundaries and interfaces of EC2 polytropes where $(n_i, n_j) = (0, 0)$ and $(\theta_{b,i}, \theta_{b,j}) = (0, 0)$, related to cases 1-7 listed in Tables 1-2. The top left panel is the source of Fig.1. Case 7 is also shown in reduced scale on the bottom right panel. The analytical continuation of the interface within the noncommon region is shown by dashed curves. With regard to the inner subsystem, the analytical continuation of open boundaries (where centrifugal support is exceeded) is shown by dotted lines. See text for further details.